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MST210

Mathematical methods, models and modelling

Book E

Cover image: This shows arrows representing a vector field and a corresponding contour map. The vector field shows the direction of change for a model of two competing populations of animals. You will meet this model in Unit 12.

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Unit 15

Vector calculus

Introduction

In Unit 2 you saw how a physical quantity can be categorised as either a scalar or a vector, and we spent some time adding, scaling and multiplying vectors. In this unit we consider scalars and vectors that have different values at different points in a region of space. For instance, you know from practical experience that the temperature in a bowl of hot soup varies quite significantly from the middle to the edge of the bowl. The temperature depends on where in the soup it is measured. Temperature is a scalar quantity, and the temperature distribution is an example of a *scalar field*. Similarly, the force of the Earth's gravity on a body in space depends on where the body is in relation to the Earth. The direction of the force is always towards the Earth's centre, and the magnitude of the force depends on the distance of the body from the Earth's centre. This dependence of force on position is an example of a *vector field*. Recall that you have already met vector fields in Subsection 1.2 of Unit 12, where they were introduced as a way of describing systems of differential equations.

For bodies close to the Earth's surface, we assume that the magnitude of the acceleration due to gravity is constant.

The main focus of this unit, and the next, is the differential calculus of scalar and vector fields, that is, the study of how scalar and vector fields vary from one point to another. Many physical laws are expressed in terms of the spatial variations of fields. One important example is the flow of heat by conduction. You will see that we can describe the spatial pattern of heat flow in a conducting material as a vector field. The rate of heat flow at any point is proportional to the negative of the temperature *gradient*. You will see in Section 3 how we can express this relationship in vector form.

Section 1 is a brief introduction to the properties of orthogonal matrices that we need in Sections 3 and 4.

Section 2 introduces pictorial representations of scalar and vector fields, such as contour curves and vector field lines, and describes how we use functions of two or three spatial variables to represent, or model, scalar and vector fields mathematically. In Section 3 we extend the discussion of the *gradient function* of a scalar field from Unit 7.

While most of the calculations are carried out using Cartesian coordinates, it is often easier to use other coordinate systems that are more suited to the symmetry of the problem. Section 4 introduces *cylindrical* and *spherical coordinate systems* for specifying points in three dimensions. We find expressions for the gradient function in each of these systems.

These coordinate systems are generalisations of the plane polar coordinate system.

The *gradient function* and the associated vector operator 'del' are fundamental tools of vector calculus: they are used to solve a range of physical and engineering problems, especially problems involving heat flow, fluid flow, dynamics and electromagnetism.

You will meet the gradient function again in Unit 16, in connection with 'conservative' forces.

1 Orthogonal matrices

Later in the unit we will study some very useful coordinate systems in three dimensions. However, utility comes at a price, and some of the calculations involved in using these systems can be quite complicated. Fortunately, much of the complication can be removed by judicious use of matrix algebra, so in this short first section we study the properties of a particular type of matrix.

One way of defining a set of coordinates is by specifying three unit vectors to form a right-handed set. The set of Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ is one example, but there are other possibilities. Recall that you can think of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as, respectively, the unit vectors in the directions of increasing x, y and z. Similarly, for the polar coordinates, we can introduce two orthogonal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Figure 1(a) shows these vectors at a point P. In the same way that $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the directions of increasing x and increasing y, respectively, $\mathbf{e}_r$ is a unit vector in the direction of increasing r (the **radial direction**), and $\mathbf{e}_\theta$ is a unit vector in the direction of increasing θ (the **tangential direction**). Although the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ have constant magnitude (they are *unit* vectors), their directions depend on the position of P. This is illustrated in Figure 1(b), which shows two different positions of P (both positions are on a circle).

Two vectors are **orthogonal** if they are perpendicular to each other.

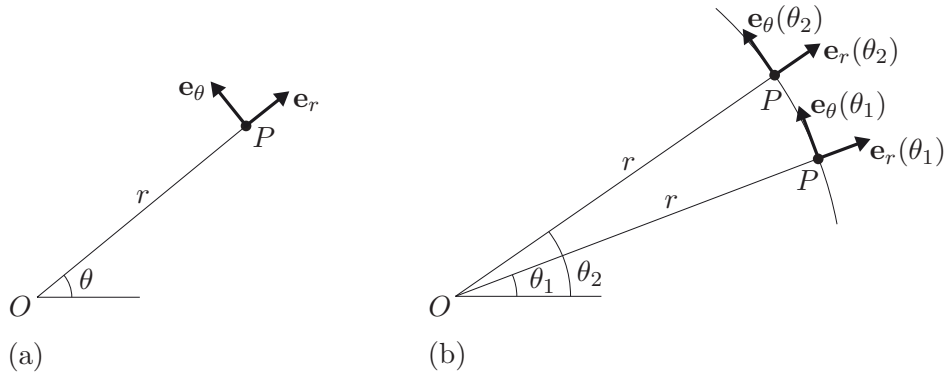


Figure 1 (a) Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ at (r, θ) in polar coordinates. (b) Unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ at two points (r, θ_1) and (r, θ_2) on a circle of radius r.

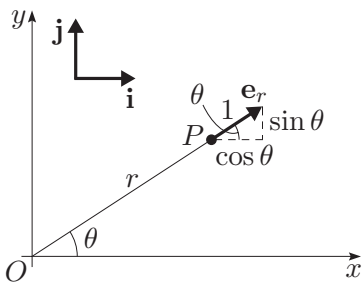


Figure 2 Unit vector $\mathbf{e}_r$ at point P with polar coordinates (r, θ)

In order to relate the coordinates of a point in one system to those in another, it will be useful to have a systematic way of transforming the corresponding sets of unit vectors. Thus we need to express each of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ as a linear combination of $\mathbf{i}$ and $\mathbf{j}$. Take the Cartesian coordinate system to have its origin at O, the centre of a circle with circumference passing through a point P, and take the positive x-axis to lie along the line from O from which the polar coordinate θ is measured, as shown in Figure 2.

First, let us consider $\mathbf{e}_r$. From Figure 2, the x-component of the *unit* vector $\mathbf{e}_r$ is $\cos\theta$, whereas its y-component is $\sin\theta$, so

$$\mathbf{e}_r = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}. \tag{1}$$

Exercise 1

Find an expression for the unit vector $\mathbf{e}_\theta$ in terms of $\mathbf{i}$, $\mathbf{j}$ and θ.

In Exercise 1 you showed that

$$\mathbf{e}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}. \tag{2}$$

Equations (1) and (2) show that $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are functions of the polar coordinate angle θ.

Exercise 2

Show explicitly that the vectors

$$\mathbf{e}_r = (\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j} \quad \text{and} \quad \mathbf{e}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}$$

form an orthogonal pair of unit vectors, that is, show that they satisfy the conditions

$$\mathbf{e}_r \cdot \mathbf{e}_r = 1, \quad \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1, \quad \mathbf{e}_r \cdot \mathbf{e}_\theta = 0.$$

To summarise, we have found that for polar coordinates,

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j}.$$

These equations can be written in matrix form as

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix},$$

This is a new use of vectors, in which the components of vectors are vector quantities.

and doing so enables us to see quickly how to express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$:

$$\begin{aligned} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}. \end{aligned}$$

It is an interesting fact that the inverse of the matrix is the same as its transpose.

Of course, matrices can be thought of in another context, as you saw in Unit 4: any 3×3 matrix represents a linear transformation of space. However, the type of transformation with which we are concerned here is rather special. A general linear transformation will alter lengths and angles, so the images of a right-handed set of unit vectors will usually be neither of unit length nor mutually perpendicular. In changing coordinates we wish to ensure that a right-handed set of unit vectors remains as such (which, incidentally, rules out reflections).

That is, a linear transformation fixes the origin and transforms lines into lines.

This means that the distance between any two points is preserved.

It may help to think of the two sets of unit vectors as having arrowheads that lie on the surface of a unit sphere (of radius 1) – see Figure 3. It is not hard to see (though we do not prove it) that any transformation taking one set to the other is simply a rotation of three-dimensional space about the origin.

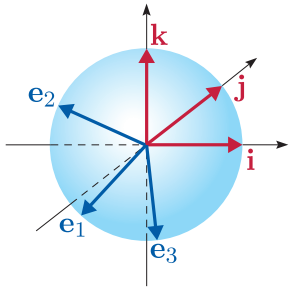


Figure 3 Two sets of orthogonal unit vectors: here **i**, **j** and **k** form one right-handed set of unit vectors, and $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ form another

So we concentrate on rotations. Consider a pair of points P_1 and P_2, and their rotated images P_1' and P_2'. A rotation does not change the distance between two points, so the distance between P_1 and P_2 is the same as the distance between P_1' and P_2'. A similar remark applies to the angle between two lines – think of the angle between two straight lines l_1 and l_2. After a rotation, the images l_1' and l_2' will be inclined at the same angle to one another. In summary, a rotation preserves distance and angle. One consequence of this observation is that since

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos\theta,$$

where θ is the angle between **a** and **b**, and since a rotation preserves each term on the right, a rotation must also preserve the dot product.

Suppose that the rotation is represented by the matrix **A**. Now take any two vectors **v** and **w**, and form their dot product $\mathbf{v} \cdot \mathbf{w}$. Since this is preserved by the rotation, we have $\mathbf{v} \cdot \mathbf{w} = (\mathbf{Av}) \cdot (\mathbf{Aw})$. Therefore

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &= \mathbf{Av} \cdot \mathbf{Aw} \\ &= (\mathbf{Av})^T(\mathbf{Aw}) \\ &= \mathbf{v}^T(\mathbf{A}^T\mathbf{A})\mathbf{w} \\ &= \mathbf{v} \cdot (\mathbf{A}^T\mathbf{Aw}). \end{aligned}$$

Recall from Unit 4 that for any two matrices, $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Because this is true for any vector **v**, it must hold when **v** is, in turn, **i**, **j** and **k**. Thus the components of $\mathbf{A}^T\mathbf{Aw}$ are the same as those of **w**, so $\mathbf{A}^T\mathbf{Aw} = \mathbf{w}$. This is true for any vector **w**, so we are forced to conclude that $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, the identity matrix. This confirms what we saw above in one (two-dimensional) case, namely that the inverse of the transformation matrix is its transpose.

A matrix **A** that satisfies the property $\mathbf{A}^T\mathbf{A} = \mathbf{I}$ is called **orthogonal**. Any orthogonal matrix must preserve lengths and angles, since (according to the above argument in reverse) it preserves the dot product.

Exercise 3

What is the determinant of an orthogonal matrix $\mathbf{A}$?

Exercise 4

Which of the following matrices are orthogonal?

(a) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ (e) $\begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}$

In Exercise 3 you saw that the determinant of an orthogonal matrix is ± 1. However, the converse is not true: a matrix with determinant 1 may not be orthogonal, as you saw in Exercise 4(a).

If the determinant of an orthogonal matrix is -1, then the matrix represents a rotated reflection, though we do not prove this.

Exercise 5

Construct a 3×3 matrix $\mathbf{A}$ that is not orthogonal but is such that $\det \mathbf{A} = 1$.

2 Scalar and vector fields

In this section we introduce scalar and vector fields, and use ideas from Unit 7 to show how they can be described mathematically.

2.1 Scalar fields

Figure 4 shows the steady-state temperature, in degrees Celsius, at different points on a square plate. Temperature is a scalar quantity, and its value depends on where on the plate it is measured. The distribution of temperatures on the plate is an example of a *scalar field*. If we define Cartesian coordinate x- and y-axes as shown, then the variable Θ, representing the temperature, will be a function of x and y, that is, $\Theta = \Theta(x, y)$. (This is an example of a function of more than one variable, as introduced in Unit 7.)

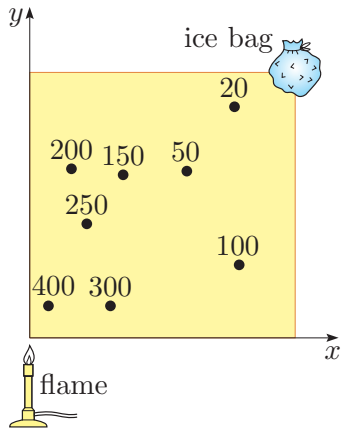


Figure 4 Steady-state temperature marked at various points on an unevenly heated plate

Another example of a scalar field is the atmospheric pressure distribution, in millibars, at ground level over the British Isles at midday on a certain day, as shown in Figure 5. Atmospheric pressure is a scalar quantity, and its value depends on where in the British Isles it is measured. The curves in Figure 5 are *isobars* and join up places where the atmospheric pressure at ground level is the same. Each isobar is labelled with the value of the atmospheric pressure in millibars. The isobars in Figure 5 give an overall picture of the atmospheric pressure across the country.

On weather maps, the pressure is usually given in millibars. The SI unit of pressure is the pascal (Pa). (1 bar $= 10^3$ millibars $= 10^5\,\text{Pa} = 10^5\,\text{N}\,\text{m}^{-2}$.)

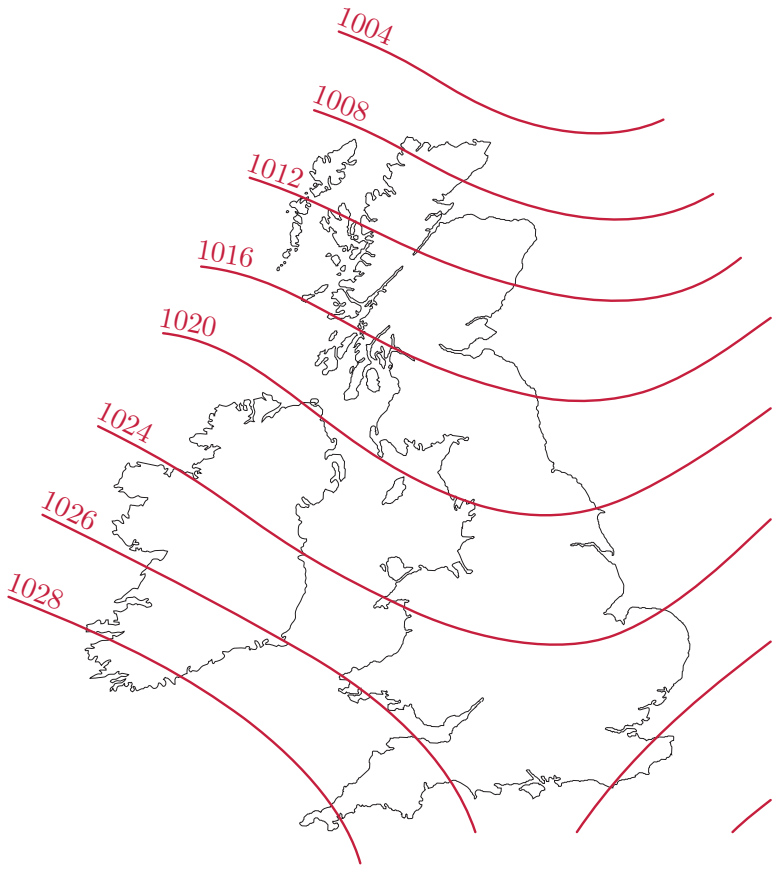


Figure 5 Isobars at ground level over the British Isles at midday

Note that the curvature of the Earth's surface is being ignored here.

If we choose London as an origin and set up a Cartesian coordinate system with the x-axis pointing East and the y-axis pointing North, then each place in the British Isles can be represented by its coordinates (x, y). The atmospheric pressure distribution can then be represented by a function of two variables $P = P(x, y)$. Of course, atmospheric pressure varies over time. But in this unit we consider only the spatial variations of scalar fields, not the time variations.

In each of the above cases, the variable representing the scalar quantity Θ or P is a function of the two coordinates x and y. These scalar distributions are examples of *two-dimensional scalar fields*, that is, they are scalar fields on a two-dimensional domain.

The steady-state temperature distribution in a room is an example of a *three-dimensional scalar field*. (The atmospheric pressure at a particular instant in time is a three-dimensional scalar field, since it varies with altitude as well as with position. In Figure 5 only the ground-level pressures at midday are shown, so this is effectively a two-dimensional scalar field.)

Each point in the room, which can be located by three coordinates, has a temperature.

In any scalar field we have:

- some scalar quantity (e.g. temperature)
- a region of a plane or a region of space, i.e. a domain, over which the scalar is defined (e.g. a flat plate or the interior of a room)
- the values of the scalar quantity at all points of the region.

The mathematical model of a two-dimensional scalar field is a function $f(x, y)$ of two spatial variables. For a three-dimensional field, a function $f(x, y, z)$ of three spatial variables is required.

Functions of two and three variables were introduced, and discussed extensively, in Unit 7.

Scalar field

A **scalar field** is a distribution of scalar values in a two- or three-dimensional region and is represented mathematically by a function of two or three spatial variables, respectively.

The domain of the field is the given two-dimensional or three-dimensional region in which the field exists. We will use the terms *scalar field* and *function of two (or three) variables* interchangeably, according to whether we want to emphasise the physical or the mathematical properties.

For example, the domain of the atmospheric pressure field is the map of the British Isles.

Example 1

The light intensity I in the region outside a spherical lamp of radius a, such as a light bulb, is inversely proportional to the square of the distance r from the centre of the lamp. The intensity 1 m away from the centre is I_0.

Specify a scalar field, as a function of three Cartesian variables, that models the light intensity in the space outside the lamp, and state the domain of the function.

Solution

The light intensity at a point a distance $r > a$ from the centre of the lamp is $I \propto 1/r^2$, with $I = I_0$ when $r = 1$. Thus $I = I_0/r^2$. Using a Cartesian coordinate system with origin at the centre of the lamp, we have $r = (x^2 + y^2 + z^2)^{1/2}$, so the scalar field function is

$$I(x, y, z) = \frac{I_0}{x^2 + y^2 + z^2} \quad ((x^2 + y^2 + z^2)^{1/2} > a).$$

The domain of the function is the statement in parentheses, expressing the fact that the function describes light intensity in the region outside the lamp.

The intensity *inside* the lamp would be represented by a different function.

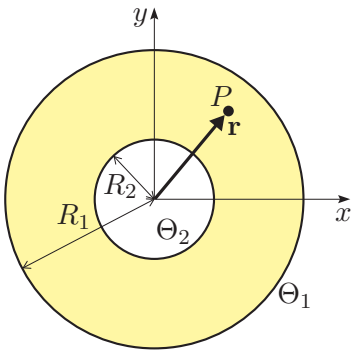


Figure 6 Annular sheet of metal with outer and inner edges kept at constant temperatures Θ_1 and Θ_2, respectively

Exercise 6

A thin circular sheet of metal of radius R_1 (in metres) has a concentric hole of radius R_2 (in metres) cut out of it (see Figure 6). The outer perimeter is maintained at a constant temperature Θ_1, and the inner perimeter is maintained at a constant temperature Θ_2. At any point P on the sheet, specified by a position vector $\mathbf{r}$ measured from the centre, the temperature (in kelvins) is given by

$$\Theta = \Theta_1 + \frac{\Theta_2 - \Theta_1}{\ln(R_2/R_1)} \ln\left(\frac{|\mathbf{r}|}{R_1}\right).$$

(a) Choose a Cartesian coordinate system with origin at the centre of the circular metal sheet, and hence specify a function of x and y that models the temperature. Describe the region of the plane over which the function is defined, and state the domain.

(b) Express the function found in part (a) in polar coordinates (r, θ).

2.2 Contour curves and contour surfaces

The scalar fields in the previous subsection were presented in three different ways. For the first scalar field, of temperature on a plate as given in Figure 4, we represented the field by showing the values of the temperature at a few points on the plate. For the second scalar field, the atmospheric pressure in the British Isles as shown in Figure 5, we represented the field by isobars (lines joining points at which the atmospheric pressure is the same) on a map of the British Isles. In both cases, the representation of the scalar field is not complete because, for instance, there is a temperature associated with *every* point of the flat plate, and not just the points shown in Figure 4. Third, in Exercise 6 and Example 1, the scalar fields were represented by functions of two or three Cartesian variables.

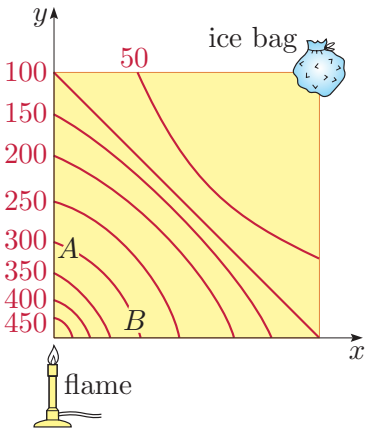


Figure 7 Isotherms on an unevenly heated flat plate

Of these modes of representation, the isobar example gives a good way of visualising the field pictorially. Figure 7 shows the temperature distribution (in degrees Celsius) over an unevenly heated flat plate similar to that shown in Figure 4. However, instead of giving the values of the temperature at different points as we did before, we have drawn curves through those points on the plate where the temperature is the same. Of course we cannot show all possible curves because that would require a curve through every point on the plate, so we choose certain values of the temperature. Then the curves give us a pictorial idea of the scalar field. So, for example, for each point on the curve AB in Figure 7, the temperature is 300°C.

For a temperature field, the curves are called *isotherms*, and in this example we have drawn the isotherms for multiples of 50°C. In one part of the plate the isotherms are much closer together than those on the rest of the plate. The temperature is changing more rapidly in the region where the isotherms are close together.

As was shown in Unit 7, this pictorial representation for pressure and temperature can be extended to more general two-dimensional scalar fields. Suppose that we have a two-dimensional scalar field $f(x, y)$, defined over a region R of the (x, y)-plane. We can represent this scalar field pictorially by drawing curves in the (x, y)-plane through those points that have equal values of f. Figure 8, for example, shows four such curves, where C_1, C_2, C_3 and C_4 are constants. The curves that we have drawn are called *contour curves*. (Contour curves were introduced in Unit 7.) Each contour curve is labelled by a scalar value.

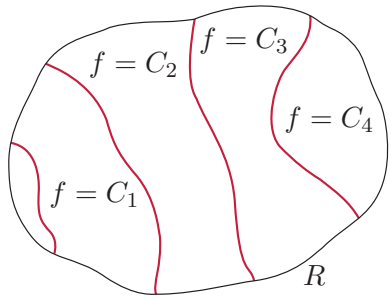


Figure 8 Contours of a function f over a region R

The isobars in a pressure field are contour curves of equal pressure, and isotherms in a temperature field are contour curves of equal temperature.

For example, consider the scalar field

$$f(x, y) = x^2 + y^2$$

defined over the rectangular domain $-1 \leq x \leq 1$, $-2 \leq y \leq 2$ (see Figure 9). The contour curves are defined by

$$f(x, y) = \text{constant},$$

or, in this case,

$$x^2 + y^2 = \text{constant}.$$

These curves are circles centred on the origin and are drawn as parts of circles within the rectangular region for which f is defined. Figure 9 shows the contour curves

$$f(x, y) = 1, \quad f(x, y) = 2, \quad f(x, y) = 3, \quad f(x, y) = 4.$$

Again we cannot draw *all* possible curves, only some of them.

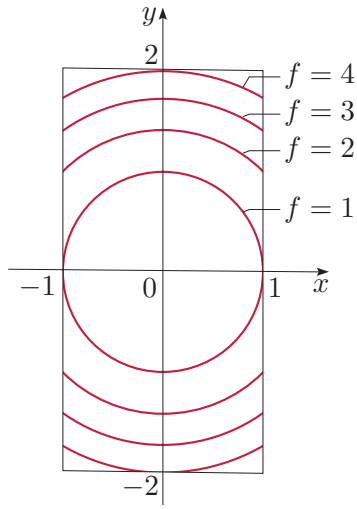


Figure 9 Contours of the function $f(x, y) = x^2 + y^2$

Contour curves

The family of curves given by $f(x, y) = C$, for different values of the constant C, are the **contour curves** of the two-dimensional scalar field f.

Exercise 7

Sketch the contour curves $f(x, y) = \frac{1}{2}$, $f(x, y) = -\frac{1}{4}$ and $f(x, y) = 1$ of the scalar field $f(x, y) = xy$, defined over the domain $x^2 + y^2 \leq 4$.

You are probably familiar with the word 'contour' from Ordnance Survey or other topographical maps, where the contour curves join points that are at the same height above sea level. The contours can give us information about the shape of the land. Figure 10 provides an example where the contour curves join points at the specified number of metres above sea level.

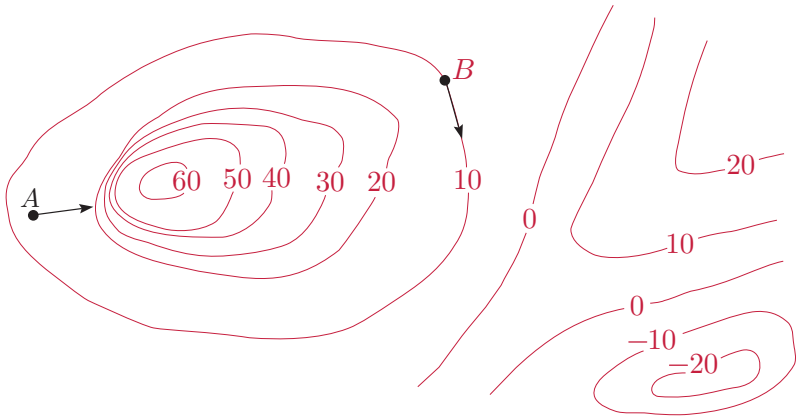


Figure 10 Topographical map showing contours indicating height above sea level

Exercise 8

Consider Figure 10.

(a) Where would you expect to find (i) a hill and (ii) a lake?

(b) Where is the land fairly level?

(c) Imagine that you set off from A and walk in the direction of the arrow. Would you expect an easy walk or a hard climb?

Exercise 8 illustrates an important feature of a scalar field that can be deduced from its contour curves. For any scalar field, the change in value of the field depends on the direction in which you move. For example, if you set off from B in Figure 10 and walk in the direction of the arrow, then initially the field values will not change (you are contour walking). However, for movement as in Exercise 8(c), which is approximately perpendicular to the contours, the scalar field changes (increases or decreases) with changing position.

So far in this subsection we have considered two-dimensional fields. Now we turn our attention to three-dimensional scalar fields.

We can represent a three-dimensional scalar field $f(x, y, z)$ in an analogous way. If we join up points for which the three-dimensional scalar field $f(x, y, z)$ has the same constant value C, then we obtain the *surface* with equation $f(x, y, z) = C$.

Contour surfaces

The family of surfaces given by $f(x, y, z) = C$, for different values of the constant C, are the **contour surfaces** of the scalar field f.

Contour surfaces give a pictorial representation of a three-dimensional scalar field. For example, the contour surfaces of the scalar field $I = I_0/(x^2 + y^2 + z^2)$, given in Example 1, are concentric spheres with centres at the origin. Two of the contour surfaces, for $I = I_0/4$ and $I = I_0/9$, are shown in Figure 11.

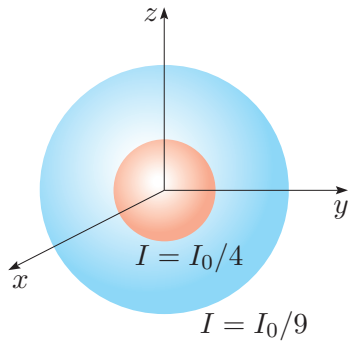


Figure 11 Contour surfaces of the function $I = I_0/(x^2 + y^2 + z^2)$

Exercise 9

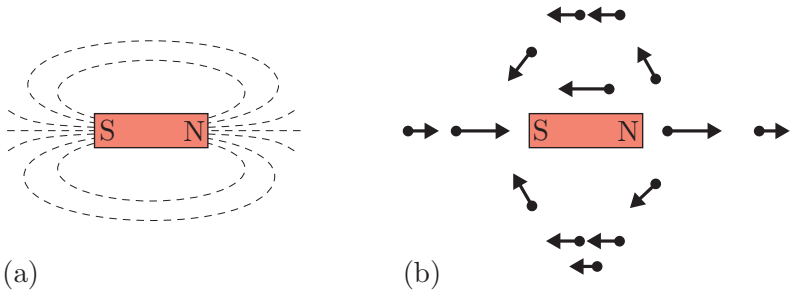
The density σ of air is $1.205\,\mathrm{kg\,m^{-3}}$ at sea level. Assume that σ decreases exponentially with altitude and is reduced to $(1.205/e)\,\mathrm{kg\,m^{-3}}$ at an altitude of 9.5×10^3 m.

Assuming that the Earth's surface is flat, choose a Cartesian coordinate system and hence express the density σ as a scalar field $\sigma(x, y, z)$. What shape are the contour surfaces?

Applied mathematicians and physicists often need to know how a quantity 'e-folds', that is, they need to know the distance (or time) in which the quantity reduces to $1/e$ of its original value.

2.3 Vector fields

Figure 12(a) shows the pattern produced when iron filings are placed near a bar magnet. The iron filings align themselves in a symmetric pattern. The same pattern is revealed if we place small compasses at various points near the magnet. We find that the arrow on each compass aligns itself in the same direction as the iron filings. The arrow shows the direction of the magnetic field at each point. The magnitude of the field can also be measured, and is found to decrease as we move away from the magnet.



(a) (b)

Figure 12 (a) Iron filing pattern near a bar magnet. (b) Magnetic field near a bar magnet.

The lengths and directions of the arrows in Figure 12(b) represent the magnitudes and directions of the magnetic field due to the bar magnet at various points on a plane through the magnet. There is a unique vector called the *magnetic field vector* at each point in the region of space around the magnet, and these vectors form a three-dimensional *vector field*.

We assume that the flow is steady. It is not changing with time.

Figure 13 shows an example of a two-dimensional vector field. The arrows represent the surface velocity at various points on a river.

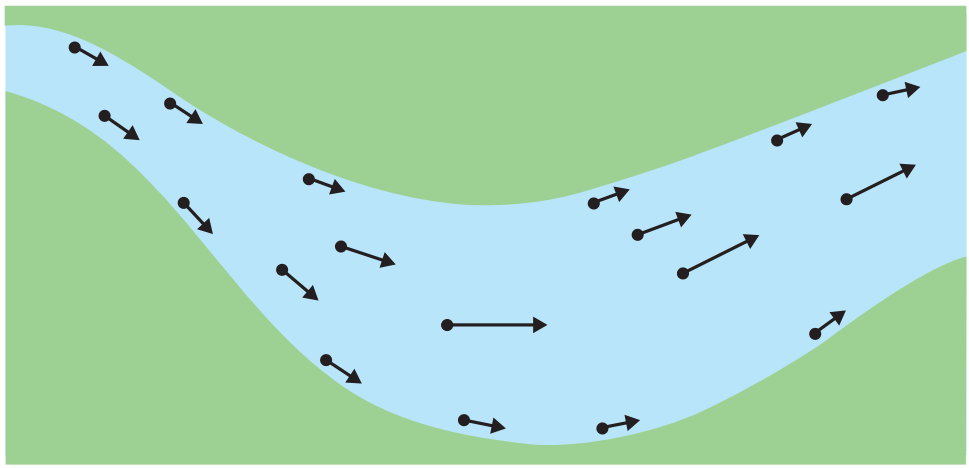


Figure 13 Surface velocity at various points along a river

The surface velocity at a particular point is the velocity that a small floating object, such as a leaf, would have at that point. Near the river banks, the water is almost at rest, so the arrows are shorter in length (indicating a smaller speed near the edges of the river). At each point on the river surface there is a unique velocity, thus defining a two-dimensional vector field of velocity vectors on the surface. If we were interested in the velocity below the surface as well, then we would need to consider the velocity vectors existing at points in the three-dimensional region forming the river itself, and this would be a three-dimensional vector field.

In any vector field we have:

- some vector quantity (e.g. a surface velocity)
- a region of a plane or a region of space, i.e. a domain, over which the vector is defined (e.g. the surface of a river)
- the magnitudes and directions of the vectors at all points in the region.

We can model a two-dimensional vector field mathematically by introducing a vector function $\mathbf{F}$ such that a vector $\mathbf{F}(x, y)$ is defined at each point (x, y) of a two-dimensional domain. In three dimensions, $\mathbf{F}(x, y, z)$ is a vector at each point (x, y, z) in a three-dimensional domain.

You met the idea of a vector field in Unit 12.

Vector field

A **vector field** is a distribution of vectors in a two- or three-dimensional region and is represented mathematically by a vector function of two or three spatial variables.

The domain of the vector function is the given region of a plane or of space in which the field exists. We use the terms *vector field* and *vector function*

interchangeably, depending on whether we want to emphasise the physical or mathematical properties. Often we use the name of the physical quantity itself, such as *force field*, *velocity field* or *magnetic field*.

Example 2

The Earth's gravitational field is an example of a vector field in space. The gravitational force on a body (i.e. the weight of the body) on or above the Earth's surface at a distance r from the centre of the Earth has magnitude inversely proportional to r^2 and is directed towards the Earth's centre.

Specify the vector field as a function of three Cartesian variables, with the domain over which it is defined.

Solution

The vector quantity is the gravitational force $\mathbf{F}$ acting on the body due to the Earth's gravity. We introduce a Cartesian coordinate system with origin O at the centre of the Earth. The magnitude $F = |\mathbf{F}|$ is proportional to $1/r^2$, so

$$F(x, y, z) = \frac{C}{r^2} = \frac{C}{x^2 + y^2 + z^2},$$

for some constant C. The gravitational force on a body at P (see Figure 14) is $-F\,\widehat{\mathbf{r}}$, where $\widehat{\mathbf{r}}$ is a unit vector in the direction from O to P. Now $\widehat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{i} + z\mathbf{k}$ is the position vector of P, so

$$\widehat{\mathbf{r}}(x, y, z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}.$$

Thus the vector function that we need is

$$\mathbf{F}(x, y, z) = -F(x, y, z)\,\widehat{\mathbf{r}}(x, y, z) = \frac{-C(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}}.$$

The domain is the region over which the vector field $\mathbf{F}$ is defined, that is, the region of space on or outside the Earth's surface, specified by

$$(x^2 + y^2 + z^2)^{1/2} \geq R,$$

where R is the radius of the (spherical) Earth.

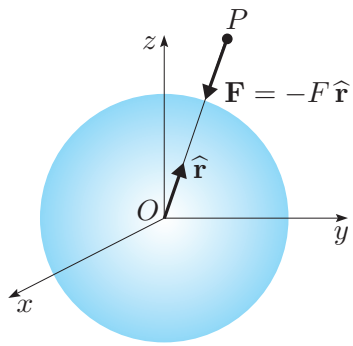


Figure 14 Gravitational force $\mathbf{F}$ acting at point P above the surface of the Earth, which is centred at the origin

At the surface of the Earth, where the magnitude of the acceleration due to gravity is measured as g, this field can be approximated by the constant-magnitude vector field $\mathbf{F} = -mg\,\widehat{\mathbf{r}}$, where m is the mass of the body, so $C = mgR^2$.

The components of a vector field $\mathbf{F}$, relative to a given Cartesian coordinate system, are themselves *scalar* fields defined over the same region as $\mathbf{F}$. Thus if $\mathbf{F}$ is a three-dimensional vector field defined on a domain D and

$$\mathbf{F}(x, y, z) = F_1(x, y, z)\,\mathbf{i} + F_2(x, y, z)\,\mathbf{j} + F_3(x, y, z)\,\mathbf{k},$$

then, separately, the three components $F_1(x, y, z) = \mathbf{F} \cdot \mathbf{i}$, $F_2(x, y, z) = \mathbf{F} \cdot \mathbf{j}$ and $F_3(x, y, z) = \mathbf{F} \cdot \mathbf{k}$ are scalar fields defined on D. For example, the x-component of the vector field $\mathbf{F}$ in Example 2 is the scalar field

$$F_1(x, y, z) = \mathbf{F} \cdot \mathbf{i} = \frac{-Cx}{(x^2 + y^2 + z^2)^{3/2}} \quad ((x^2 + y^2 + z^2)^{1/2} \geq R).$$

Alternatively, we could use the column vector notation of Unit 4 and write a vector field in three dimensions as

$$\mathbf{F}(x, y, z) = \begin{pmatrix} F_1(x, y, z) \\ F_2(x, y, z) \\ F_3(x, y, z) \end{pmatrix}.$$

Exercise 10

The weight **F** of a body of mass m inside the Earth (e.g. in a mine shaft) has a magnitude given by mgr/R, where r is the distance from the Earth's centre and R is the Earth's radius. The direction of **F** is towards the Earth's centre.

Express **F** as a vector function $\mathbf{F}(x, y, z)$ using the same Cartesian coordinate system as in Example 2. What is the weight of the body at the Earth's centre?

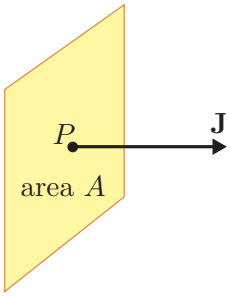


Figure 15 Heat flow **J** at point P in a conductor

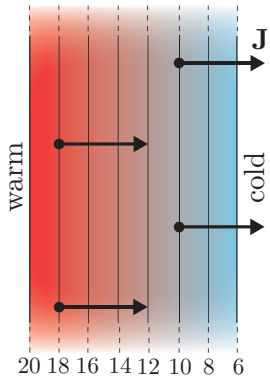


Figure 16 Cross-section of a wall in a house showing heat flow (arrows) and temperature contours

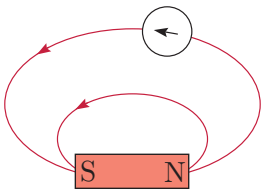


Figure 17 Magnetic field lines of a bar magnet

The pattern of heat flow in a heat-conducting material is another example of a vector field. Heat is a form of energy (and is therefore measured in joules) that flows through a conducting body due to the motion and interactions of the constituent particles (e.g. molecules) of the body. When the magnitude and direction of the heat flow vary from place to place in a heat conductor, the heat flow rate per unit area at a point P in the conductor is a vector **J** whose direction is the direction of the heat flow at P. The magnitude of **J** is the rate at which heat flows across a very small plane surface containing P and oriented at right angles to the direction of flow, divided by the area A of the surface (see Figure 15). So $|\mathbf{J}|$ is the heat flow rate per unit area at P and has units of joules per second per square metre (i.e. $\mathrm{J\,s^{-1}\,m^{-2}}$, or $\mathrm{W\,m^{-2}}$, since 1 watt (W) equals $1\,\mathrm{J\,s^{-1}}$). (The vector **J** is called the *heat flow vector* or the *heat conduction vector*.) To define $|\mathbf{J}|$ at P formally, we take the limit as the area A of the plane surface approaches zero, that is,

$$|\mathbf{J}| = \lim_{A \to 0} \frac{\text{heat flow rate across surface area } A}{A}.$$

Figure 16 shows a selection of arrows representing a heat flow vector field **J** in a cross-section of a wall of a heated house. Shown also are vertical lines representing the scalar temperature contours. These lines are the intersections of the temperature contour surfaces with the plane of the figure. The contours are labelled by the temperature, in degrees Celsius, for the case where the inside surface is at 20°C and the outside surface is at 6°C. Hence Figure 16 depicts two fields: a scalar (temperature) field and a vector (heat flow) field. It is assumed that the temperature has a different constant value on each face of the wall and that the wall is homogeneous. Then the vector **J** has the same magnitude and direction everywhere inside the wall.

If we draw continuous curves in the domain such that at any point, the tangent to the curve is parallel to the direction of the vector field at that point, then the curves are called the **vector field lines** of the vector field. For instance, the orientation of iron filings near a bar magnet suggests a family of continuous curves. Two such curves are shown in Figure 17. The arrow on a vector field line specifies the sense, that is, the direction along the line, of the vector field. If we draw a curve showing the path of a leaf floating on the surface of a river, then at each point on the path, the

tangent to the path is parallel to the direction of the velocity vector of the leaf. This line is a vector field line for the velocity vector field. If we were to show the vector field lines of the heat flow vector **J** in Figure 16, they would be at right angles to the temperature contour surfaces shown as lines in that figure.

Exercise 11

Figure 18 shows the cross-section of a long cylindrical central heating pipe carrying hot water. The heat flow vector **J** in the metal of the pipe is given (in the steady state) by

$$\mathbf{J}(x, y, z) = \frac{C(x\mathbf{i} + y\mathbf{j})}{x^2 + y^2} \quad (a \leq \sqrt{x^2 + y^2} \leq b),$$

where a, b and C are positive constants, and the z-axis is the central axis of the pipe.

(a) Specify the x-component of **J**.

(b) Describe the vector field lines of **J**.

(c) How does the magnitude $|\mathbf{J}|$ vary with distance $\rho = \sqrt{x^2 + y^2}$ from the z-axis?

(d) Consider an imaginary cylindrical surface of length h with its axis on the z-axis and radius ρ such that $a \leq \rho \leq b$. By making use of your answers to parts (b) and (c), show that the rate at which heat flows outwards across the whole curved surface of this section of the cylinder is independent of the radius ρ. What does this result mean in terms of conservation of heat energy?

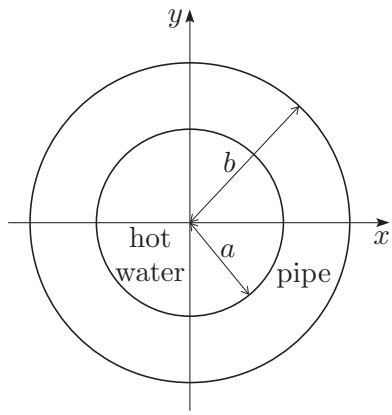


Figure 18 Cross-section of a central heating pipe

Note that although the vector field **J** of Exercise 11 exists in three dimensions, its z-component is zero and there are no variations in the z-direction. The field could therefore be represented by a two-dimensional vector function

$$\mathbf{J}(x, y) = \frac{C(x\mathbf{i} + y\mathbf{i})}{x^2 + y^2} \quad (a \leq \sqrt{x^2 + y^2} \leq b).$$

Exercise 12

When a long thin straight wire carries a uniform surface distribution of static positive electric charge, the electric field vector **E** at any point outside the wire has magnitude $|\mathbf{E}|$ that is inversely proportional to the perpendicular distance from the centre of the wire, and direction pointing directly away from the wire.

Introduce a Cartesian coordinate system and hence specify the vector field $\mathbf{E}(x, y, z)$. Take $|\mathbf{E}|$ to be the constant E_0 at unit distance from the wire. Describe the vector field lines.

Describing the wire as 'thin' implies that it has negligible radius. It is just a straight line.

3 Gradient of a scalar field

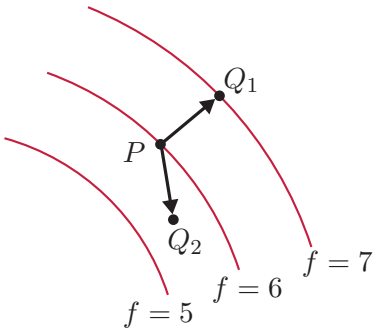


Figure 19 Contours of a function f

An important property of any field is the way in which the field value changes from one point to another. These spatial variations can be quite complicated even for a scalar field, since the change of field value in going from a point P to a nearby point Q may depend on the direction as well as the magnitude of the displacement $\overrightarrow{PQ}$. Consider Figure 19, which shows three contour curves of a scalar field f. The change in f is $7 - 6 = 1$ when the displacement $\overrightarrow{PQ_1}$ is made, and about $5.5 - 6 = -0.5$ for a displacement $\overrightarrow{PQ_2}$ of the same magnitude as $\overrightarrow{PQ_1}$ but in a different direction. We will show in this section that the way in which a scalar field varies in *any* direction can be found from the *gradient function*, which was introduced in Unit 7. There is a gradient function at each point in a scalar field, so the gradient function is a vector field.

In Unit 7 you saw how to use the gradient function to find the magnitude of the steepest upward slope at a point on a surface (and the direction in which that steepest slope occurs). The gradient function for a general two-dimensional scalar field $f(x, y)$ was expressed in terms of partial derivatives of f. You also saw in Unit 7 how the gradient function can be used to calculate slopes in arbitrary directions. In Subsections 3.1 and 3.2 we review those ideas, and in Subsection 3.3 we consider how the expression for the gradient function can be generalised to three dimensions.

You met del in Unit 7, in conjunction with the gradient function.

Subsection 3.4 discusses the *vector differential operator* called 'del', which plays a unifying role in vector calculus. Finally, in Subsection 3.5, we see how to define del in polar coordinates.

3.1 Gradient function

A two-dimensional scalar field $f(x, y)$ is defined only in the (x, y)-plane. This leaves the z-axis of the coordinate system free for showing the scalar values f, that is, we can put $z = f(x, y)$. Thus the set of points $(x, y, f(x, y))$ is a surface above (or below) the (x, y)-plane, giving a graphical picture of the scalar field. In general, the surface $(x, y, f(x, y))$ is a graphical representation of the scalar field f and shows how f varies with position in the (x, y)-plane.

You saw examples of such surfaces in Unit 7.

Consider a general two-dimensional scalar field f and a path in the (x, y)-plane parametrised by the arc length s – that is, as s varies, the point $(x(s), y(s))$ moves along the path in the (x, y)-plane. The height z of the surface will also vary as s varies, that is, $z = z(s)$. The rate of change of z with s at a point P in the (x, y)-plane is the slope or steepness of the surface for movement in the direction of the path, that is, it is the vertical rise of z over a small horizontal change in arc length δs in the direction of the path at P, and is given by the chain rule, which we can write as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}, \tag{3}$$

where all the derivatives and partial derivatives are to be evaluated at P.

Let us first consider a small change in length along the path at P in the (x, y)-plane. Since it is small, we can take it to be in the direction of the tangent to the path at P, a direction given by a unit vector $\widehat{\mathbf{d}}$, say, as shown in Figure 20. If α is the angle at P between $\widehat{\mathbf{d}}$ and the positive direction of the x-axis, then $\cos\alpha = dx/ds$ and $\sin\alpha = dy/ds$, and equation (3) may be rewritten as

$$\frac{dz}{ds} = \frac{\partial f}{\partial x}(a,b)\cos\alpha + \frac{\partial f}{\partial y}(a,b)\sin\alpha, \tag{4}$$

where we have shown explicitly that the partial derivatives are to be evaluated at the point P whose coordinates are (a, b). While the angle α depends on the direction of the path at P, the partial derivatives in equation (4) are independent of any particular path through P, and depend only on the scalar field f and the coordinates (a, b) of P. We can take advantage of this by using the **gradient function** in the (x, y)-plane at the point (a, b):

$$\mathbf{grad}\, f(a,b) = \frac{\partial f}{\partial x}(a,b)\,\mathbf{i} + \frac{\partial f}{\partial y}(a,b)\,\mathbf{j}.$$

The unit vector $\widehat{\mathbf{d}}$ may be written in terms of α as

$$\widehat{\mathbf{d}} = (\cos\alpha)\,\mathbf{i} + (\sin\alpha)\,\mathbf{j}.$$

So we can now write equation (4) as the dot product of $\mathbf{grad}\, f(a, b)$ and $\widehat{\mathbf{d}}$:

$$\frac{dz}{ds} = \mathbf{grad}\, f(a,b)\cdot\widehat{\mathbf{d}} = |\mathbf{grad}\, f(a,b)|\cos\theta, \tag{5}$$

where θ is the angle between $\mathbf{grad}\, f$ and $\widehat{\mathbf{d}}$, as shown in Figure 21.

You can see from equation (5) that dz/ds, the slope or steepness of the surface (the graph of f), has its maximum value (when $\cos\theta = 1$) for movement parallel to the (x, y)-plane in a direction $\widehat{\mathbf{d}}$ that coincides with the direction of the gradient function. Then we have

$$\left(\frac{dz}{ds}\right)_{\max} = |\mathbf{grad}\, f(a,b)|.$$

Hence the direction of the gradient function at (a, b) shows the direction of maximum slope of the surface, and the magnitude of the gradient function is equal to the magnitude of the maximum slope.

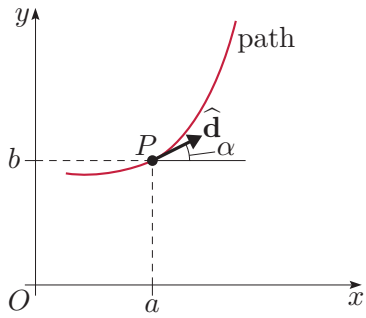


Figure 20 Unit vector at the point P

In Unit 7 you met the alternative notation ∇f for $\mathbf{grad}\, f$, but we will postpone using that notation until we have defined ∇ in Subsection 3.4.

As with vectors, in handwritten work, **grad** should be denoted with a straight or wavy underline.

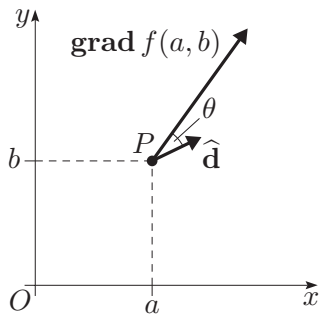


Figure 21 Finding the dot product of $\mathbf{grad}\, f(a, b)$ and $\widehat{\mathbf{d}}$

Maximum derivative of a scalar field

The value of the **maximum derivative** of the scalar field $f(x, y)$ at the point (a, b) is given by $|\mathbf{grad}\, f(a, b)|$, and the **direction** of this maximum derivative is given by the unit vector

$$\frac{\mathbf{grad}\, f(a,b)}{|\mathbf{grad}\, f(a,b)|}.$$

There is such a vector at each point in the (x, y)-plane, so $\mathbf{grad}\, f(x, y)$ is a vector field, often written simply as $\mathbf{grad}\, f$. In terms of the Cartesian unit vectors, we define it as follows.

Gradient in two dimensions

The **gradient in Cartesian coordinates** (x, y) **of a two-dimensional scalar field** f is the vector field

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}. \tag{6}$$

This definition is the same as that given in Unit 7, but expressed in terms of fields. As the gradient is a vector field, i.e. a vector function, it is sometimes referred to as the *gradient function*.

Thus to find the gradient function at a particular point, it is necessary to find the two partial derivatives at that point and substitute them into equation (6).

The function f in the definition can represent any two-dimensional scalar field. For example, suppose that $\Theta(x, y)$ is the temperature field of a flat plate in the (x, y)-plane. Then $\mathbf{grad}\,\Theta$ is a vector parallel to the plane of the plate pointing in the direction in which the temperature increases most rapidly with distance, and $|\mathbf{grad}\,\Theta|$ is the value of the maximum rate of temperature change with distance.

For a temperature field measured in kelvins, the slope has units of kelvins per metre, i.e. $\mathrm{K\,m^{-1}}$.

The gradient function of a two-dimensional scalar field $f(x, y)$ is always parallel to the (x, y)-plane. It is a vector having only $\mathbf{i}$ and $\mathbf{j}$ components. You can think of it as lying in the (x, y)-plane, or you may prefer to imagine it positioned at the point $(x, y, f(x, y))$ and lying in a plane parallel to the (x, y)-plane. Equation (5) shows that we can use the gradient function to work out the slope, or steepness, dz/ds of a scalar field for movement in any direction parallel to the (x, y)-plane. This slope is the *derivative* of f in the *specified direction*.

Derivative of a scalar field

The **derivative of a scalar field** f in a direction specified by a unit vector $\widehat{\mathbf{d}}$ is given by

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}}.$$

This derivative is often referred to as a *directional derivative* or *slope*, the latter being particularly apt for a two-dimensional scalar field.

3.2 Calculating gradients

We can calculate gradient vectors of two-dimensional scalar fields $f(x, y)$, and derivatives in specified directions, by using the two definitions above.

Throughout this unit, when calculating $\mathbf{grad}\, f$ we assume that f is differentiable on its domain (in two and three dimensions).

Example 3

Evaluate **grad** f at the point $(0, 1)$ when $f(x, y) = \ln(x + 2y)$. Determine the magnitude of **grad** f and the unit vector in the direction of **grad** f at the point $(0, 1)$. What is the derivative of f in the direction of $\mathbf{i} + \mathbf{j}$ at $(0, 1)$?

Solution

The vector field **grad** f is found from the first partial derivatives of f:

$$\frac{\partial f}{\partial x} = \frac{1}{x + 2y}, \quad \frac{\partial f}{\partial y} = \frac{2}{x + 2y}.$$

So from equation (6),

$$\mathbf{grad}\, f = \left(\frac{1}{x + 2y}\right)\mathbf{i} + \left(\frac{2}{x + 2y}\right)\mathbf{j}.$$

This is the gradient vector field.

Putting $x = 0$ and $y = 1$, we have

$$\mathbf{grad}\, f(0, 1) = \tfrac{1}{2}\mathbf{i} + \mathbf{j}.$$

So the magnitude of **grad** f at $(0, 1)$ is

$$|\mathbf{grad}\, f(0, 1)| = \left(\tfrac{1}{4} + 1\right)^{1/2} = \sqrt{5}/2.$$

The unit vector in the direction of **grad** f at $(0, 1)$ is

$$\frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = \frac{\mathbf{i} + 2\mathbf{j}}{\sqrt{5}}.$$

The unit vector $\widehat{\mathbf{d}}$ in the direction of $\mathbf{i} + \mathbf{j}$ is given by

$$\widehat{\mathbf{d}} = \frac{\mathbf{i} + \mathbf{j}}{|\mathbf{i} + \mathbf{j}|} = \frac{\mathbf{i} + \mathbf{j}}{\sqrt{2}}.$$

Hence the derivative of f in the direction of $\mathbf{i} + \mathbf{j}$ is given at $(0, 1)$ by the dot product

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}} = \frac{3}{2\sqrt{2}}.$$

Exercise 13

Evaluate **grad** f at the point $(-1, 2)$ when $f(x, y) = x^2 y$. Find the derivative of f in the x-direction at $(-1, 2)$.

Exercise 14

A two-dimensional scalar field has the form $f(x, y) = (x^2 + y^2)^{1/2}$.

(a) Find the vector field **grad** f, and evaluate **grad** f at the point $(1, 1)$.

(b) Specify the unit vector $\widehat{\mathbf{d}}$ at $\frac{\pi}{6}$ to the positive x-direction and at $\frac{\pi}{3}$ to the positive y-direction. What is the derivative of f in the direction of $\widehat{\mathbf{d}}$ at $(1, 1)$?

Exercise 15

Determine the vector field **grad** g, where $g(x, y) = x^2y - y^2x$. Find the magnitude and direction of the steepest slope on the surface $z = g(x, y)$ at the point $(1, 1)$.

Example 4

Evaluate **grad** f at the point $(1, 2)$ when $f(x, y) = x^2 + y^2$. Show that the direction of **grad** f at $(1, 2)$ is normal to the tangent line to the contour curve of f at the point $(1, 2)$.

Solution

The first partial derivatives of f and their values at the point $(1, 2)$ are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial x}(1, 2) = 2,$$
$$\frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial y}(1, 2) = 4.$$

Thus from equation (6),

$$\textbf{grad}\, f(1, 2) = \frac{\partial f}{\partial x}(1, 2)\,\mathbf{i} + \frac{\partial f}{\partial y}(1, 2)\,\mathbf{j} = 2\mathbf{i} + 4\mathbf{j}.$$

The contour curves are obtained by setting f equal to a constant and are circles centred on the origin. The contour curve through the point $(1, 2)$ is shown in Figure 22, together with the tangent line and the normal to the tangent line through $(1, 2)$. The position vector of $(1, 2)$ is $\mathbf{i} + 2\mathbf{j}$, and this is normal to the tangent line. But **grad** $f(1, 2) = 2\mathbf{i} + 4\mathbf{j} = 2(\mathbf{i} + 2\mathbf{j})$. Hence **grad** $f(1, 2)$ is normal to the tangent to the contour curve at $(1, 2)$.

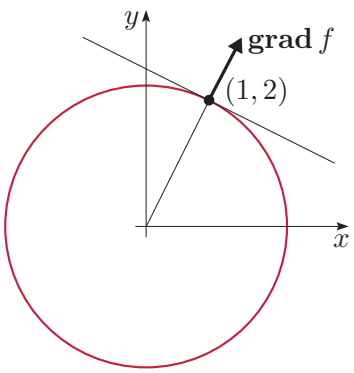


Figure 22 The contour of the function $f(x, y) = x^2 + y^2$ that passes through the point $(1, 2)$

As was shown in Unit 7, it is true generally that the gradient function at a point is normal to the tangent line to the contour curve at that point. This is clear from Figure 23, which shows a contour curve $f(x, y) = C$ passing through a point P, and a unit vector $\widehat{\mathbf{e}}$ on the tangent line at P. The scalar field f does not change along the contour curve, so the derivative of f in the direction tangential to the contour curve is zero, and we must have **grad** $f \cdot \widehat{\mathbf{e}} = 0$ at P. Hence **grad** f must be normal to $\widehat{\mathbf{e}}$.

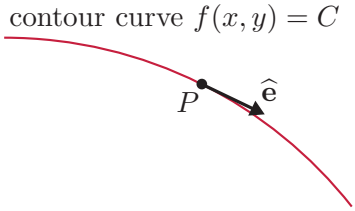


Figure 23 A contour curve

Exercise 16

Find a vector whose direction is normal to the curve $x^2 - 2xy + y^2 = 9$ at the point $(0, 3)$.

(*Hint*: The curve is a contour of the scalar field $f(x, y) = x^2 - 2xy + y^2$.)

3.3 Gradient function in three dimensions

You have seen that for a two-dimensional scalar field $f(x, y)$, we can use the z-axis for showing the scalar values $z = f(x, y)$ and so construct the surface $(x, y, f(x, y))$. When we have a three-dimensional scalar field $f(x, y, z)$, such as the temperature distribution in a room, all three spatial coordinates (x, y, z) are needed to specify the domain of the function, and there is no 'fourth spatial dimension' for showing the function values. However, we can still consider a parametrised path in space and use the chain rule in three dimensions to give the rate of change of f along the path. Proceeding as before, we find that the derivative of $f(x, y, z)$ in the direction of $\widehat{\mathbf{d}}$, a unit vector tangential to the path, parametrised by the arc length s at a point (a, b, c) is

$$\frac{df}{ds} = \mathbf{grad}\, f(a, b, c) \cdot \widehat{\mathbf{d}},$$

where the three-dimensional vector

$$\mathbf{grad}\, f(a, b, c) = \frac{\partial f}{\partial x}(a, b, c)\,\mathbf{i} + \frac{\partial f}{\partial y}(a, b, c)\,\mathbf{j} + \frac{\partial f}{\partial z}(a, b, c)\,\mathbf{k}$$

is the gradient vector of f at the point (a, b, c).

There is a gradient vector at each point in the domain of f, so $\mathbf{grad}\, f$ is a vector field in three dimensions.

Gradient and derivative in three dimensions

The **gradient in Cartesian coordinates** (x, y, z) **of a three-dimensional scalar field** f is the vector field

$$\mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}. \tag{7}$$

The **derivative** of f in a direction specified by a unit vector $\widehat{\mathbf{d}}$ is given by

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}}.$$

This derivative may also be referred to as the *directional derivative*.

In three dimensions, the gradient vector at a point is always normal to the tangent plane to the contour surface passing through that point.

Example 5

Consider the scalar field

$$f(x, y, z) = -C/(x^2 + y^2 + z^2)^{1/2},$$

where C is a positive constant.

Show that the gravitational vector field $\mathbf{F} = -C\,\hat{\mathbf{r}}/r^2$, where $\hat{\mathbf{r}}$ is a unit vector in the direction of the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, can be expressed in terms of f by the relationship $\mathbf{F} = -\mathbf{grad}\, f$.

The vector field $\mathbf{F}$ was considered in Example 2. In that case, the complete specification of the scalar field f would include a statement of the domain $(x^2 + y^2 + z^2)^{1/2} \geq R$, where R is the radius of the Earth. We often omit the domain statement for convenience.

Solution

We have

$$f(x, y, z) = \frac{-C}{(x^2 + y^2 + z^2)^{1/2}},$$

so

$$\frac{\partial f}{\partial x} = \frac{Cx}{(x^2 + y^2 + z^2)^{3/2}},$$
$$\frac{\partial f}{\partial y} = \frac{Cy}{(x^2 + y^2 + z^2)^{3/2}},$$
$$\frac{\partial f}{\partial z} = \frac{Cz}{(x^2 + y^2 + z^2)^{3/2}}.$$

Thus from equation (7),

$$\mathbf{grad}\, f = \frac{C(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{(x^2 + y^2 + z^2)^{3/2}} = \frac{C\,\hat{\mathbf{r}}}{x^2 + y^2 + z^2}.$$

Hence $-\mathbf{grad}\, f$ is the vector function describing the gravitational vector field $\mathbf{F} = -C\,\hat{\mathbf{r}}/r^2$.

In Example 5, a vector field $\mathbf{F}$ is a scalar multiple of the gradient of a scalar field f. Scalar and vector fields that are used to model physical quantities in the real world are often related in this way. A similar relationship holds between the heat flow vector field $\mathbf{J}$ and the gradient function of a temperature field Θ in a heat-conducting material. This relationship, known as **Fourier's law**, is given by

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta, \tag{8}$$

where κ is the thermal conductivity of the material. Equation (8) has a natural physical interpretation. We are used to the notion that in a conducting material, heat flows from points of higher temperature to points of lower temperature. Thus we would expect the heat flow vector field $\mathbf{J}$ to point in the direction where the temperature field Θ decreases most rapidly. This is given by Fourier's law as expressed in equation (8).

As with two-dimensional scalar fields, $\mathbf{grad}\,\Theta$ points in the direction where Θ increases most rapidly, therefore $-\mathbf{grad}\,\Theta$ points in the direction where Θ *decreases* most rapidly.

Exercise 17

Evaluate **grad** Θ at the point $(-1, 1, 0)$ for the temperature field

$$\Theta(x, y, z) = A - B \ln\left(\frac{\sqrt{x^2 + y^2}}{a}\right),$$

where A, B and a are positive constants, $\sqrt{x^2 + y^2} \geq a$ and $0 < a < \sqrt{2}$.

A scalar field of this form can be used to model the temperature distribution in the wall of a conducting cylindrical pipe with its axis along the z-axis and inner radius a (as in Exercise 11). The pipe carries hot water at temperature A.

You have seen examples in which a vector field is equal to a constant multiple of the gradient of a scalar field. However, note that not all vector fields can be obtained as gradients of scalar fields.

We will return to this topic in Unit 16.

Exercise 18

Consider $f(x, y, z) = x^2y^2z^2$.

(a) Find the maximum value of the derivative of f at the point $(-1, 1, 1)$, and the unit vector that specifies the direction in which this maximum occurs.

(b) Find the derivative of f at the point $(2, 1, -1)$ in the direction of $3\mathbf{i} + 4\mathbf{k}$.

3.4 Gradient as a vector operator

We can write the partial derivative $\partial f/\partial x$ of a scalar field f as $(\partial/\partial x)f$, where the symbol $\partial/\partial x$ is a **differential operator**. The operator acquires meaning only when given an **operand** on its right-hand side, that is, a function f on which to operate. Thus the differential operator $\partial/\partial x$ acts on a scalar function f to give another scalar function $\partial f/\partial x$.

Consider the quantity $\mathbf{i}\,\partial/\partial x$. This, too, is a differential operator. It acts on a scalar function f to give the vector function $(\mathbf{i}\,\partial/\partial x)f = \mathbf{i}\,\partial f/\partial x$. For example, when $f(x, y, z) = x^3 - 2xy^2 + z$, we have the vector function

$$(\mathbf{i}\,\partial/\partial x)f = \mathbf{i}\,\partial f/\partial x = \mathbf{i}(3x^2 - 2y^2).$$

The operator $\mathbf{i}\,\partial/\partial x$ is an example of a **vector differential operator**.

Consider the expression for **grad** f in equation (7). We can write it as

$$\mathbf{grad}\, f = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) f, \tag{9}$$

where the quantity in brackets is a vector differential operator commonly known as **del** (or **nabla** in some texts) and denoted by the bold symbol $\boldsymbol{\nabla}$. (The symbol $\boldsymbol{\nabla}$ was introduced in Unit 7. Recall that in handwritten work it should be denoted with a straight or wavy underline.)

The del operator

The vector differential operator **del** is denoted by $\boldsymbol{\nabla}$ and in Cartesian coordinates is given by

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{10}$$

We used the alternative notation for the gradient function in two dimensions in Unit 7. In two dimensions the third component of $\boldsymbol{\nabla}$ is redundant, and we have

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y}.$$

We can now write the gradient function of f as **grad** f or as $\boldsymbol{\nabla} f$, that is,

$$\textbf{grad}\, f = \boldsymbol{\nabla} f.$$

This use of del may appear to be nothing more than a notational device. However, as you will see later, this vector differential operator takes on the significance of a powerful unifying concept in vector calculus.

Example 6

Find the vector field $\boldsymbol{\nabla} f$, where f is the scalar field

$$f(x, y, z) = x^2 - 2xz.$$

Solution

$\boldsymbol{\nabla} f$ is the same as **grad** f. Hence, using equation (9),

$$\begin{aligned}\boldsymbol{\nabla} f &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(x^2 - 2xz)\\ &= \mathbf{i}(2x - 2z) + \mathbf{j}(0) + \mathbf{k}(-2x)\\ &= 2(x - z)\mathbf{i} - 2x\mathbf{k}.\end{aligned}$$

Exercise 19

Consider the scalar field $f(x, y, z) = e^{-a(x^2+y^2)-bz}$, where a and b are positive constants. Show that

$$\boldsymbol{\nabla} f = -(2a(x\mathbf{i} + y\mathbf{j}) + b\mathbf{k})f,$$

and evaluate $\boldsymbol{\nabla} f$ at the origin. Find the x-component of $\boldsymbol{\nabla} f(1, 2, 3)$.

Notice that we can use matrix notation to write the operator del as

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z}\end{pmatrix}. \tag{11}$$

This idea is useful when we need a formula for del in other coordinate systems, as you will see in the next subsection.

3.5 Gradient function in polar coordinates

It is sometimes more convenient to express a scalar field in terms of polar coordinates (r, θ). (One example is when the field is symmetric under rotations about the origin, so it has the form $f(r)$.) In order to calculate the gradient of a field in these coordinates, we need an expression for **grad** f in polar coordinates. The first step is to recall from Section 1 how the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are related to the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ (see Figure 24):

$$\mathbf{e}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j},$$
$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{i} + \cos\theta\,\mathbf{j},$$

or, in matrix form,

$$\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix}.$$

We can express $\mathbf{i}$ and $\mathbf{j}$ in terms of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ by inverting this orthogonal matrix, obtaining

$$\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\end{pmatrix} = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\mathbf{e}_r\\ \mathbf{e}_\theta\end{pmatrix}.$$

Now we use the chain rule from Unit 7 to calculate $\partial f/\partial r$ in terms of $\partial f/\partial x$ and $\partial f/\partial y$. The form of the chain rule that we need is

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r}.$$

But since $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$\frac{\partial x}{\partial r} = \cos\theta \quad \text{and} \quad \frac{\partial y}{\partial r} = \sin\theta.$$

Hence

$$\frac{\partial f}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y},$$

and since this is true for any function f, we have

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}.$$

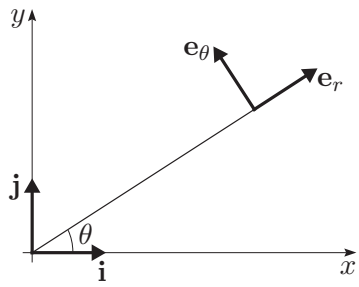


Figure 24 Illustrating the geometrical relationship between the Cartesian unit vectors $\mathbf{i}$ and $\mathbf{j}$ and the polar unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$

Exercise 20

Derive a similar expression for $\partial/\partial\theta$ in terms of r, $\cos\theta$, $\sin\theta$, $\partial/\partial x$ and $\partial/\partial y$.

These two pieces of information may be summarised in matrix form as

$$\begin{pmatrix}\frac{\partial}{\partial r}\\ \frac{\partial}{\partial\theta}\end{pmatrix} = \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\end{pmatrix}.$$

Although the matrix is not orthogonal (because of the presence of r), it can be inverted to give

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix}.$$

Now we can find the formula for del in polar coordinates:

$$\begin{aligned}
\boldsymbol{\nabla} &= \begin{pmatrix} \mathbf{i} & \mathbf{j} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix} \\
&= \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix} \right)^T \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}^T \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -\frac{1}{r}\sin\theta \\ \sin\theta & \frac{1}{r}\cos\theta \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \end{pmatrix}^T \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{pmatrix} \\
&= \mathbf{e}_r \frac{\partial}{\partial r} + \frac{1}{r}\mathbf{e}_\theta \frac{\partial}{\partial \theta}.
\end{aligned}$$

Gradient function in polar coordinates

The **gradient function in polar coordinates** of a scalar field f is

$$\mathbf{grad}\, f = \mathbf{e}_r \frac{\partial f}{\partial r} + \mathbf{e}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta}, \tag{12}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors in the r and θ directions.

Exercise 21

Find the gradient vector at a point (r, θ) for each of the following scalar functions.

(a) $f(r,\theta) = r^2 - 2r\cos\theta$ (b) $f(r,\theta) = \dfrac{e^{-r\sin\theta}}{r^2}, \quad r > 0$

4 Three-dimensional polar coordinate systems

Often the first step in solving a problem is to choose an appropriate coordinate system, one that will express the problem in the simplest possible form. This may involve choosing the origin and the orientation of

a Cartesian coordinate system. On the other hand, there are many problems for which other types of coordinate system are more appropriate. For example, you will see in Unit 20 how useful a polar coordinate system can be for problems involving circular motion.

This section introduces two non-Cartesian coordinate systems, which are especially useful for representing three-dimensional fields that have cylindrical or spherical symmetry. Such symmetries are usually easy to recognise. A three-dimensional scalar field has **cylindrical symmetry** when the field values depend only on the distance from a fixed straight line. If we call this line the z-axis, then the field depends only on the single variable $\rho = \sqrt{x^2 + y^2}$. The scalar field of Exercise 17 is an example of a cylindrically symmetric field. A three-dimensional scalar field has **spherical symmetry** when the field values depend only on the distance from a fixed point, which may be taken as the origin. The light intensity field I of Example 1, which varies only with $r = \sqrt{x^2 + y^2 + z^2}$, is an example of a spherically symmetric field.

4.1 Cylindrical coordinates

A cylindrical coordinate system extends the familiar polar coordinates (r, θ) to three dimensions. The coordinate r is now labelled ρ, since it represents the distance from the z-axis (rather than from the origin), and the coordinate θ is now labelled ϕ for consistency with spherical coordinates, as will become clear in Subsection 4.2. The z-coordinate of the Cartesian coordinate system provides the third variable, to give the following system.

Some texts continue to use r in place of ρ and/or θ in place of ϕ.

Cylindrical coordinates

Any point P can be represented by the triple (ρ, ϕ, z), where z is the distance from P to the (x, y)-plane, and (ρ, ϕ) are the polar coordinates of the projection N of P onto the (x, y)-plane (see Figure 25). Cylindrical coordinates are related to the Cartesian coordinates (x, y, z) by

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z, \tag{13}$$

$$\rho = (x^2 + y^2)^{1/2}, \quad \cos\phi = x/\rho, \quad \sin\phi = y/\rho. \tag{14}$$

We require that

$$\rho \geq 0, \quad -\pi < \phi \leq \pi, \quad z \in \mathbb{R}.$$

The value of ϕ for points on the z-axis ($\rho = 0$) is undefined. By convention, we put $\phi = 0$ for such points.

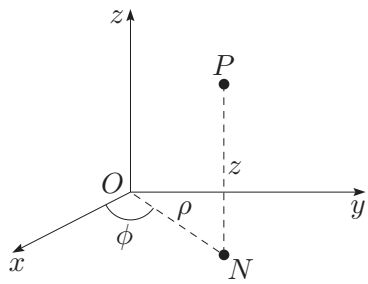
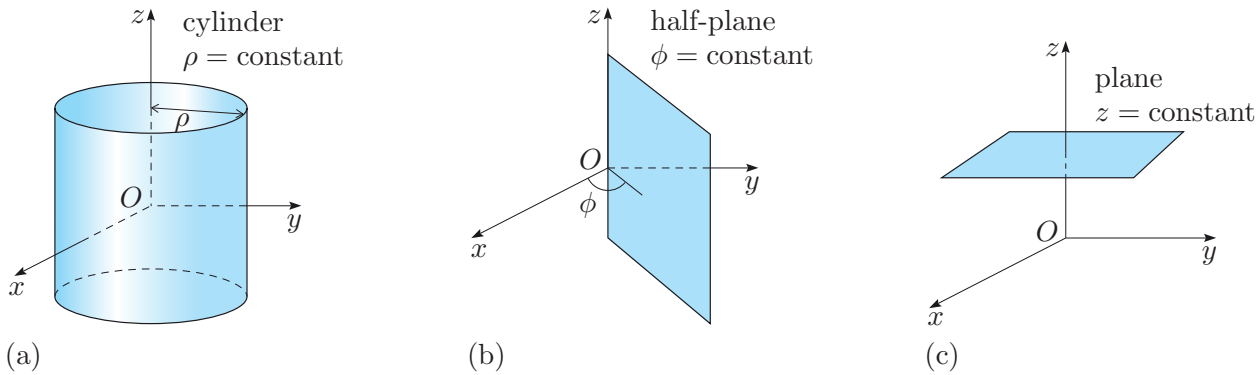


Figure 25 Cylindrical coordinates

The convention in this module is that the angle ϕ is measured anticlockwise from the x-axis and is in the range $-\pi < \phi \leq \pi$. In some texts the range of ϕ is taken as $0 \leq \phi < 2\pi$.

In three dimensions we use the symbol ρ for the distance $\sqrt{x^2 + y^2}$ from the z-axis. The symbol r is reserved for the distance $\sqrt{x^2 + y^2 + z^2}$ from the origin.

In Cartesian coordinates, the surfaces $x =$ constant, $y =$ constant and $z =$ constant are planes. In cylindrical coordinates, the surfaces $\rho =$ constant (> 0) are circular cylinders with axis the z-axis and radius ρ (see Figure 26(a)). The surfaces $\phi =$ constant $(\neq 0)$ are half-planes that do not contain the z-axis (see Figure 26(b)). By convention, only the half-plane $\phi = 0$ contains the z-axis on its boundary. The surfaces $z =$ constant are planes perpendicular to the z-axis (see Figure 26(c)).



(a) (b) (c)

Figure 26 (a) Surface with cylindrical coordinate $\rho =$ constant. (b) Surface with cylindrical coordinate $\phi =$ constant. (c) Surface with cylindrical coordinate $z =$ constant.

Example 7

(a) Determine the cylindrical coordinates (ρ, ϕ, z) of the following points given in Cartesian coordinates (x, y, z):

$$(5, 0, 0), \quad (0, 5, 0), \quad (-3, 3, 2), \quad (1, \sqrt{3}, 3)$$

(b) Express the scalar field

$$f(x, y, z) = \frac{x\left(x^2 + y^2 + 3\right)}{x^2 + y^2} \quad (x^2 + y^2 > 0,\ z > 1)$$

in cylindrical form, and determine, if possible, the field value at the point where $\rho = 4$, $\phi = \frac{\pi}{3}$ and $z = 2$.

Solution

(a) All the z-values remain unchanged. The first two points are both at a distance 5 from the z-axis and lie in the (x, y)-plane, so $\rho = 5$ and $z = 0$ for both of them. The angle ϕ is measured in the (x, y)-plane anticlockwise from the x-axis (see Figure 25), so $\phi = 0$ for the first point and $\phi = \frac{\pi}{2}$ for the second point.

The third point is at a distance $((-3)^2 + 3^2)^{1/2} = 3\sqrt{2}$ from the z-axis, so $\rho = 3\sqrt{2}$. The angle ϕ is in the second quadrant of the (x, y)-plane and has the value $\frac{3\pi}{4}$.

More formally, we could use equations (14) to obtain

$$\rho = ((-3)^2 + 3^2)^{1/2} = 3\sqrt{2}, \quad \cos\phi = -\tfrac{3}{3\sqrt{2}}, \quad \sin\phi = \tfrac{3}{3\sqrt{2}},$$

from which $\phi = \frac{3\pi}{4}$.

Using equations (14) for the fourth point,

$$\rho = (1^2 + (\sqrt{3})^2)^{1/2} = 2, \quad \cos\phi = \tfrac{1}{2}, \quad \sin\phi = \tfrac{\sqrt{3}}{2},$$

from which $\phi = \frac{\pi}{3}$.

So the four points in cylindrical coordinates are, respectively,

$$(5, 0, 0), \quad (5, \tfrac{\pi}{2}, 0), \quad (3\sqrt{2}, \tfrac{3\pi}{4}, 2), \quad (2, \tfrac{\pi}{3}, 3).$$

(b) Using equations (13), we see that

$$x^2 + y^2 + 3 = \rho^2(\cos^2\phi + \sin^2\phi) + 3\rho^2 + 3$$

and

$$\frac{x}{x^2 + y^2} = \frac{\rho\cos\phi}{\rho^2} = \frac{\cos\phi}{\rho}.$$

Hence

$$f(\rho, \phi, z) = \frac{(\rho^2 + 3)\cos\phi}{\rho} \quad (\rho > 0,\ z > 1).$$

First we confirm that the point with cylindrical coordinates $(4, \frac{\pi}{3}, 2)$ is within the domain of the function. It clearly satisfies $\rho > 0$ and $z > 1$. No condition is explicitly placed on ϕ, so we assume that ϕ can have any value within its range. Thus the point is within the domain of f, and

$$f(4, \tfrac{\pi}{3}, 2) = \frac{(4^2 + 3)\cos\frac{\pi}{3}}{4} = \tfrac{19}{8}.$$

Exercise 22

Consider the scalar field

$$\lambda(\rho, \phi, z) = \frac{\cos(\frac{\pi}{2}z)}{\rho} \quad (\rho > 0,\ -1 \le z \le 1).$$

(a) Describe, in geometric terms, the domain of the function.

(b) Determine, where possible, the values of λ at points P, Q and R on the positive x-, y- and z-axes, respectively, each at unit distance from the origin.

(c) Determine the value of λ at all points on a circle of radius 2, with its centre on the z-axis, in the plane $z = \frac{1}{4}$.

The cylindrical coordinate system can also be used to represent vector fields. However, before we can do this, we need to specify a set of three unit vectors in cylindrical coordinates that play the same role as **i**, **j** and **k** in Cartesian systems. Figure 27(a) shows three mutually perpendicular unit vectors at a point P with cylindrical coordinates (ρ, ϕ, z), as follows.

In the polar coordinate notation (r, θ) of Section 1 and Subsection 3.5, the unit vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are $\mathbf{e}_r$ and $\mathbf{e}_\theta$, respectively.

- The unit vector $\mathbf{e}_\rho$ at P points in the direction of increasing ρ, i.e. perpendicularly away from the z-axis at P.
- The unit vector $\mathbf{e}_\phi$ at P points in the direction of increasing ϕ, i.e. tangential to the circle through P and centred at $(0, 0, z)$.
- The unit vector $\mathbf{e}_z$ is the same as the Cartesian unit vector **k**, i.e. it is in the positive z-direction.

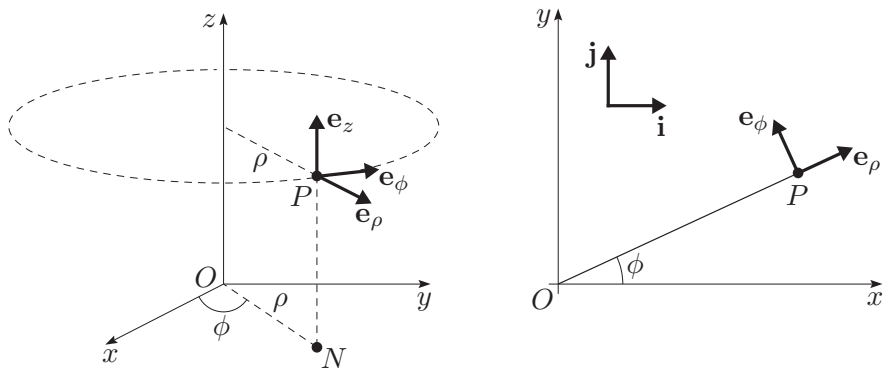


Figure 27 (a) Unit vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ lie in a plane parallel to the (x, y)-plane; $\mathbf{e}_\rho$ points away from the z-axis, $\mathbf{e}_\phi$ is perpendicular to $\mathbf{e}_\rho$ in the direction of increasing ϕ, and $\mathbf{e}_z$ is the same as the Cartesian unit vector **k**. (b) Projection of $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ onto the (x, y)-plane.

You can see from Figure 27(b) that these unit vectors are related to the Cartesian unit vectors in the following way.

Cylindrical unit vectors and Cartesian unit vectors

The cylindrical unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ can be expressed in terms of the Cartesian unit vectors **i**, **j** and **k** as

$$\mathbf{e}_\rho = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}}, \tag{15}$$

$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi = \frac{-y\mathbf{i} + x\mathbf{j}}{\sqrt{x^2 + y^2}}, \tag{16}$$

$$\mathbf{e}_z = \mathbf{k}. \tag{17}$$

We can write this information more economically in matrix form as

$$\begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}. \tag{18}$$

The matrix is orthogonal, so its inverse is its transpose:

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix}. \tag{19}$$

Unlike the Cartesian unit vectors, the cylindrical unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ are not all constant vectors, since the directions of $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ depend on the position of P. In other words, $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are non-constant vector fields.

Since the cylindrical unit vectors are mutually perpendicular at each point, we have the following definition.

Cylindrical components

A vector $\mathbf{F}$ can be expressed in terms of the cylindrical unit vectors $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ as

$$\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z, \tag{20}$$

where $F_\rho = \mathbf{F} \cdot \mathbf{e}_\rho$, $F_\phi = \mathbf{F} \cdot \mathbf{e}_\phi$ and $F_z = \mathbf{F} \cdot \mathbf{e}_z$. The scalar quantities F_ρ, F_ϕ and F_z are the **cylindrical components** of $\mathbf{F}$.

Example 8

Consider the vector field

$$\mathbf{F}(x, y, z) = \frac{C(x\mathbf{i} + y\mathbf{j})}{x^2 + y^2} \quad (a \leq \sqrt{x^2 + y^2} \leq b),$$

where a, b and C are positive constants.

(a) Express this vector field in cylindrical coordinates.

(b) Assuming that $C = 1$, $a = 2$ and $b = 4$, determine the magnitude and direction of the vector field at the point with cylindrical coordinates $(3, \pi, 7)$. What are the cylindrical components of $\mathbf{F}$ at this point?

Solution

(a) We can write

$$\mathbf{F}(x, y, z) = \frac{C}{\sqrt{x^2 + y^2}} \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}} \quad (a \leq \sqrt{x^2 + y^2} \leq b).$$

Thus using equations (14) and (15),

$$\mathbf{F}(\rho, \phi, z) = \frac{C}{\rho} \mathbf{e}_\rho \quad (a \leq \rho \leq b). \tag{21}$$

(b) From equation (21), the magnitude of the vector field at the point with cylindrical coordinates $(3, \pi, 7)$ is $|\mathbf{F}(3, \pi, 7)| = |C/\rho| = \frac{1}{3}$, and the direction of the vector field is given by $\mathbf{e}_\rho$, that is, the direction perpendicularly away from the z-axis through the point. Again from equation (21), the cylindrical components of $\mathbf{F}$ at $(3, \pi, 7)$ are $F_\rho = \frac{1}{3}$ and $F_\phi = F_z = 0$.

Exercise 23

Consider the vector field

$$\mathbf{u}(x, y, z) = \frac{(xe^{-z^2} - y)\mathbf{i} + (ye^{-z^2} + x)\mathbf{j}}{x^2 + y^2}.$$

Express $\mathbf{u}$ in cylindrical coordinates, and hence specify its cylindrical components.

Grad in cylindrical coordinates

If we are given a scalar field f in cylindrical coordinates, then we can work out the vector field **grad** f without first having to convert the cylindrical coordinates into Cartesian coordinates. In order to do this, we need an expression for **grad** f in terms of partial derivatives with respect to the cylindrical variables. We start by finding relationships between $\partial/\partial\rho$, $\partial/\partial\phi$, $\partial/\partial z$ and $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$. This is done easily using the chain rule.

First we recall the relationship between Cartesian coordinates and cylindrical coordinates from equations (13):

$$x = \rho\cos\phi, \quad y = \rho\sin\phi, \quad z = z.$$

Now applying the chain rule gives, for any function f,

$$\begin{aligned}\frac{\partial f}{\partial \rho} &= \frac{\partial x}{\partial \rho}\frac{\partial f}{\partial x} + \frac{\partial y}{\partial \rho}\frac{\partial f}{\partial y} + \frac{\partial z}{\partial \rho}\frac{\partial f}{\partial z} \\ &= \cos\phi\,\frac{\partial f}{\partial x} + \sin\phi\,\frac{\partial f}{\partial y}.\end{aligned}$$

We deduce that

$$\frac{\partial}{\partial \rho} = \cos\phi\,\frac{\partial}{\partial x} + \sin\phi\,\frac{\partial}{\partial y}.$$

Similarly, we can find an expression for $\partial/\partial\phi$:

$$\begin{aligned}\frac{\partial}{\partial \phi} &= \frac{\partial x}{\partial \phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi}\frac{\partial}{\partial y} + \frac{\partial z}{\partial \phi}\frac{\partial}{\partial z} \\ &= -\rho\sin\phi\,\frac{\partial}{\partial x} + \rho\cos\phi\,\frac{\partial}{\partial y}.\end{aligned}$$

Fortunately, there is no additional work for $\partial/\partial z$ because

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}.$$

We can write these three operator equations in matrix form as

$$\begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

This is very similar to the operator equations derived for polar coordinates in Subsection 3.5.

This matrix is not orthogonal for $\rho \neq 1$, but we can deduce that its inverse is

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\rho\sin\phi & \rho\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi & 0 \\ \sin\phi & \frac{1}{\rho}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To see this, note that the 3×3 matrix that we require to invert consists of the 2×2 matrix

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{pmatrix}$$

coinciding with the first two elements of the diagonal, with 1 being the remaining element of the diagonal, and all the remaining off-diagonal elements being zero. We can then use the method described in Unit 4 for finding the inverse of a 2×2 matrix, to obtain

This is an example of a matrix said to be in *block-diagonal* form.

$$\begin{pmatrix} \cos\phi & \sin\phi \\ -\rho\sin\phi & \rho\cos\phi \end{pmatrix}^{-1} = \begin{pmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi \\ \sin\phi & \frac{1}{\rho}\cos\phi \end{pmatrix},$$

from which the inverse of the 3×3 matrix now follows. Hence we can now express $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ in terms of $\partial/\partial\rho$, $\partial/\partial\theta$ and $\partial/\partial z$:

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos\phi & -\frac{1}{\rho}\sin\phi & 0 \\ \sin\phi & \frac{1}{\rho}\cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial \rho} \\ \frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{pmatrix}. \tag{22}$$

Recall from equation (11) that

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}.$$

Now from equation (19) we have

$$\begin{aligned} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{pmatrix} = \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}^T &= \left(\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\phi \\ \mathbf{e}_z \end{pmatrix} \right)^T \\ &= \begin{pmatrix} \mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}^T. \end{aligned}$$

Substituting for $(\partial/\partial x \quad \partial/\partial y \quad \partial/\partial z)^T$ from equation (22), we see that

$$\begin{aligned}\boldsymbol{\nabla} &= \begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z\end{pmatrix}\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}^T\begin{pmatrix}\cos\phi & -\frac{1}{\rho}\sin\phi & 0\\ \sin\phi & \frac{1}{\rho}\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial\rho}\\ \frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\cos\phi & -\frac{1}{\rho}\sin\phi & 0\\ \sin\phi & \frac{1}{\rho}\cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial\rho}\\ \frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{e}_\rho & \mathbf{e}_\phi & \mathbf{e}_z\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \frac{1}{\rho} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{\partial}{\partial\rho}\\ \frac{\partial}{\partial\phi}\\ \frac{\partial}{\partial z}\end{pmatrix}.\end{aligned}$$

Thus

$$\boldsymbol{\nabla} = \mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z}. \tag{23}$$

Gradient function in cylindrical coordinates

The **gradient function in cylindrical coordinates** of a scalar field f is

$$\mathbf{grad}\, f = \mathbf{e}_\rho\frac{\partial f}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial f}{\partial\phi} + \mathbf{e}_z\frac{\partial f}{\partial z}, \tag{24}$$

where $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ are unit vectors in the ρ-, ϕ- and z-directions, respectively.

When the scalar field has cylindrical symmetry, the scalar field does not vary with ϕ or z, so the equation simplifies to

$$\mathbf{grad}\, f = \mathbf{e}_\rho\frac{\partial f}{\partial\rho}. \tag{25}$$

Example 9

Consider the scalar field

$$f(\rho,\phi,z) = \ln\rho \quad (\rho > 0).$$

Determine the vector field $\mathbf{grad}\, f$. Specify the magnitude and direction of the gradient vector at a point that is 5 units perpendicularly away from the z-axis. Check your answer for $\mathbf{grad}\, f$ by expressing the scalar function f in Cartesian coordinates and using the Cartesian expression for the gradient function.

Solution

We can use the form for **grad** f in equation (25), since the function varies with ρ only. Hence the vector field is

$$\mathbf{grad}\, f = \mathbf{e}_\rho \frac{\partial}{\partial \rho}(\ln \rho) = \mathbf{e}_\rho \left(\frac{1}{\rho}\right) = \frac{1}{\rho}\,\mathbf{e}_\rho \quad (\rho > 0).$$

At a perpendicular distance of 5 units from the z-axis, we have $\rho = 5$, so $|\mathbf{grad}\, f| = \frac{1}{5}$. The direction of **grad** f is along the line perpendicularly out from the z-axis, passing through the point.

Since $\rho = (x^2 + y^2)^{1/2}$, in Cartesian coordinates the scalar field is

$$f(x, y, z) = \ln((x^2 + y^2)^{1/2}) = \tfrac{1}{2}\ln(x^2 + y^2).$$

Hence, from equation (7), as before,

$$\mathbf{grad}\, f = \tfrac{1}{2}\left(\frac{2x}{x^2 + y^2}\,\mathbf{i} + \frac{2y}{x^2 + y^2}\,\mathbf{j}\right) = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2} = \frac{1}{\rho}\,\mathbf{e}_\rho.$$

Exercise 24

(a) Describe the geometric shapes of the surfaces defined in a cylindrical system by the following.

(i) $\rho = 2$ (ii) $\phi = \frac{\pi}{2}$

(b) Show that the cylindrical unit vectors $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are perpendicular.

(*Hint*: Use the fact that the Cartesian unit vectors are mutually perpendicular.)

(c) Find the gradient of the scalar field $F(\rho, \phi, z) = \rho e^{-z}$ at a point on the positive x-axis that is 5 units from the origin.

Exercise 25

Consider the scalar field

$$U(x, y, z) = 1/(x^2 + y^2), \quad \text{where } x^2 + y^2 > 0.$$

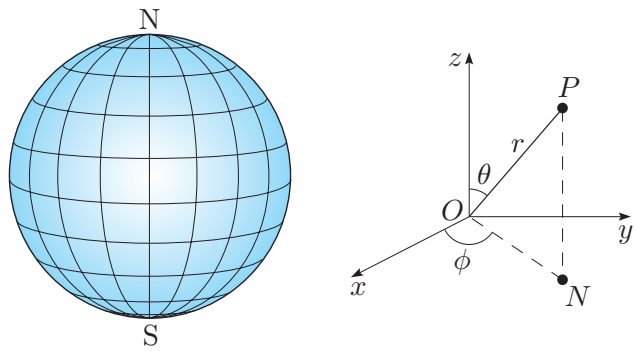
Find $\boldsymbol{\nabla} U$ using the Cartesian form of the gradient function. Express U in cylindrical coordinates, and hence confirm your answer for the gradient function. Sketch some contour surfaces of U and vector field lines for $\boldsymbol{\nabla} U$.

Exercise 26

Determine the cylindrical components of the vector field **grad** f for the scalar field $f(\rho, \phi, z) = (\cos z)/\rho$, where $\rho > 0$.

4.2 Spherical coordinates

In the spherical coordinate system, one of the three coordinates of any point is specified as the magnitude r of the position vector $\mathbf{r}$ of the point. Hence for fixed r, all points with coordinate r lie on the surface of a sphere of radius r centred on the origin. To specify position on the sphere, two angle coordinates θ and ϕ are used, based on the idea of locating a point on the Earth's surface by giving its latitude and longitude (see Figure 28(a)).



(a) (b)

Figure 28 (a) Surface of the Earth, showing curves of constant latitude and longitude. (b) Spherical coordinates r, θ and ϕ.

The **polar angle** θ is a measure of latitude on the sphere, with θ increasing from 'north pole' to 'south pole', so $\theta = 0$ for points on the positive z-axis, $\theta = \frac{\pi}{2}$ for points in the (x, y)-plane, and $\theta = \pi$ for points on the negative z-axis. The **azimuthal angle** ϕ is a measure of longitude on the sphere and is the same as the coordinate ϕ in the cylindrical coordinate system.

It may help you to think of ϕ as increasing 'west to east'; compare Figures 27(a) and 28(b).

Spherical coordinates

Any point P (see Figure 28(b)) can be represented by the triple (r, θ, ϕ), where r is the distance from P to the origin, and θ and ϕ are the polar and azimuthal angles, respectively. In Figure 28(b), N is the projection of P onto the (x, y)-plane.

The spherical coordinates of P are related to the Cartesian coordinates (x, y, z) by

$$x = ON\cos\phi = r\sin\theta\cos\phi, \tag{26}$$

$$y = ON\sin\phi = r\sin\theta\sin\phi, \tag{27}$$

$$z = r\cos\theta, \tag{28}$$

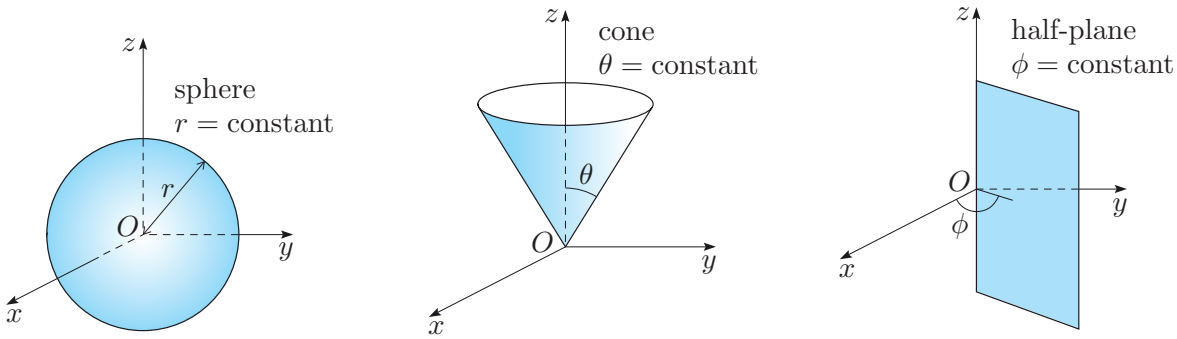
$$r = \sqrt{x^2 + y^2 + z^2}. \tag{29}$$

We require that

$$r \geq 0, \quad -\pi < \phi \leq \pi, \quad 0 \leq \theta \leq \pi.$$

There are explicit expressions for θ and ϕ in terms of the other variables, but we do not need them in this module.

The surface $r =$ constant is a sphere (see Figure 29(a)). The surface $\theta =$ constant is a cone (see Figure 29(b)), or a plane if $\theta = \frac{\pi}{2}$, or a half-line if $\theta = 0$ or $\theta = \pi$. The surface $\phi =$ constant $(\neq 0)$ is a half-plane that does not contain the z-axis (see Figure 29(c)). By convention, only the half-plane $\phi = 0$ contains the z-axis on its boundary.



(a) (b) (c)

Figure 29 (a) Surface with spherical coordinate $r =$ constant. (b) Surface with spherical coordinate $\theta =$ constant. (c) Surface with spherical coordinate $\phi =$ constant.

Exercise 27

Give the spherical coordinates of points with the following Cartesian coordinates.

(a) $(5, 0, 0)$

(b) $(0, 5, 0)$

The mutually perpendicular unit vectors for the spherical coordinate system are shown in Figure 30. They are related to the Cartesian unit vectors in the following way.

Spherical unit vectors and Cartesian unit vectors

The unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ can be expressed in terms of the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ as

$$\mathbf{e}_r = \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta, \tag{30}$$

$$\mathbf{e}_\theta = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta, \tag{31}$$

$$\mathbf{e}_\phi = -\mathbf{i}\sin\phi + \mathbf{j}\cos\phi. \tag{32}$$

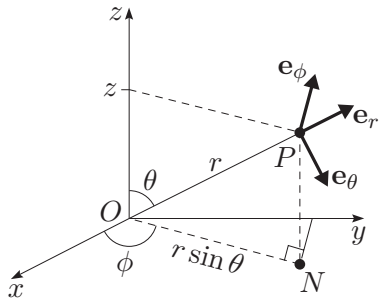


Figure 30 Orthogonal unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ lie in the plane containing the triangle ONP with $\mathbf{e}_r$ and $\mathbf{e}_\theta$ pointing in the directions of increasing r and θ, respectively. Unit vector $\mathbf{e}_\phi$ is perpendicular to the plane containing the triangle ONP and points in the direction of increasing ϕ.

The directions of the unit vectors for the spherical coordinate system at a point P depend on the θ- and ϕ-coordinates of P (but not on r). Once again, this information is more compactly expressed as a matrix equation

$$\begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}. \tag{33}$$

Since the matrix is orthogonal, its inverse is its transpose, so we have

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_r \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{pmatrix}. \tag{34}$$

Since the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are mutually perpendicular, we have the following definition.

Spherical components

A vector $\mathbf{F}$ can be expressed in terms of the spherical unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ as

$$\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi, \tag{35}$$

where $F_r = \mathbf{F}\cdot\mathbf{e}_r$, $F_\theta = \mathbf{F}\cdot\mathbf{e}_\theta$ and $F_\phi = \mathbf{F}\cdot\mathbf{e}_\phi$. The scalars F_r, F_θ and F_ϕ are the **spherical components** of $\mathbf{F}$.

Exercise 28

Express the following vector field in terms of spherical coordinates and unit vectors, and hence determine its spherical components:

$$\mathbf{F}(x,y,z) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2+y^2+z^2} \quad (x^2+y^2+z^2 > 0).$$

Grad in spherical coordinates

The derivation is similar to (but more complicated than) that for cylindrical coordinates. We omit the details here.

The gradient function can be expressed in spherical coordinates as follows.

Gradient function in spherical coordinates

The **gradient function in spherical coordinates** of a scalar field f is

$$\mathbf{grad}\, f = \mathbf{e}_r\frac{\partial f}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial f}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}, \tag{36}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors in the r-, θ- and ϕ-directions, respectively.

For spherically symmetric fields, in which the field values do not vary with θ or ϕ, the gradient simplifies to

$$\mathbf{grad}\, f = \mathbf{e}_r \frac{\partial f}{\partial r}. \tag{37}$$

Example 10

Consider the scalar field expressed in Cartesian coordinates as

$$V(x, y, z) = \frac{-I_0}{(x^2 + y^2 + z^2)^{1/2}} \quad ((x^2 + y^2 + z^2)^{1/2} > a > 0),$$

where I_0 is a positive constant.

Express this field in spherical coordinates, and hence show that $|\mathbf{grad}\, V| = I$, where I is the light intensity field specified in Example 1 as $I = I_0/(x^2 + y^2 + z^2)$.

Solution

We know that $x^2 + y^2 + z^2 = r^2$, so

By definition, $r \geq 0$.

$$V(r, \theta, \phi) = -I_0/r \quad (r > a).$$

The field is spherically symmetric, so we can use equation (37) to obtain

$$\mathbf{grad}\, V = \mathbf{e}_r \frac{\partial}{\partial r}\left(\frac{-I_0}{r}\right) = \frac{I_0}{r^2}\,\mathbf{e}_r.$$

Hence

$$|\mathbf{grad}\, V| = \frac{I_0}{r^2} = \frac{I_0}{x^2 + y^2 + z^2},$$

which is the light intensity field I of Example 1.

Exercise 29

Specify the spherical components of the vector field $\mathbf{grad}\, V$ in Example 10.

Exercise 30

(a) Describe the geometric shape of the surface defined in a spherical coordinate system by $\theta = \frac{\pi}{6}$.

(b) Use equation (30) to confirm that the spherical unit vector $\mathbf{e}_r$ is of unit magnitude.

(c) Find the gradient in spherical coordinates of the scalar field $U(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the spherical coordinate point $(5, \frac{\pi}{2}, \pi)$. Give your answer in terms of the Cartesian unit vectors.

Exercise 31

Consider the scalar field

$$V(r, \theta, \phi) = \frac{M(3\cos^2\theta - 1)}{r^3} \quad (r > 0),$$

where M is a positive constant. Three points A, B and C have the following positions: A is on the positive y-axis 3 units from the origin, B has Cartesian coordinates $(0, 0, 1)$, and C has spherical coordinates $(1, \pi, 0)$.

On the z-axis, where ϕ is not defined, by convention we have $\phi = 0$.

(a) Determine, where possible, the value of the scalar field V at the three points A, B and C.

(b) If $V(r, \theta, \phi)$ represents the potential energy of a particle at a point, then the force acting on the particle at that point is given by

$$\mathbf{F} = -\mathbf{grad}\, V.$$

This result, relating force and potential energy, is obtained in Unit 16.

Determine the magnitude and direction of the force on the particle when the particle is at A.

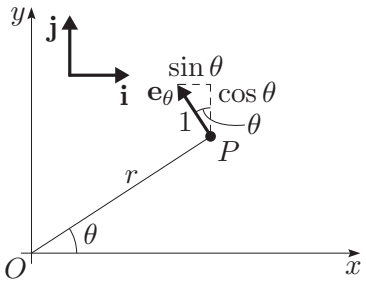
Learning outcomes

After studying this unit, you should be able to:

- appreciate the concept of a scalar field
- interpret and sketch contour curves for a given scalar field
- appreciate the concept of a vector field
- interpret and sketch vector field lines for a given vector field
- determine the gradient function of a scalar field, and calculate the derivative of a scalar field in a specified direction
- convert between the Cartesian coordinates of a point and the cylindrical or spherical coordinates of the point
- express a scalar or vector field in cylindrical or spherical coordinates, given the field in Cartesian coordinates.

Solutions to exercises

Solution to Exercise 1



From the figure above, the x-component of the unit vector $\mathbf{e}_\theta$ is $-\sin\theta$, and the y-component is $\cos\theta$. Hence

$$\mathbf{e}_\theta = (-\sin\theta)\,\mathbf{i} + (\cos\theta)\,\mathbf{j}.$$

Solution to Exercise 2

Since $\mathbf{i}$ and $\mathbf{j}$ are themselves an orthogonal pair of unit vectors, we know from Unit 2 that

$$\mathbf{i}\cdot\mathbf{i} = 1, \quad \mathbf{j}\cdot\mathbf{j} = 1, \quad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0.$$

Hence

$$\begin{aligned}
\mathbf{e}_r\cdot\mathbf{e}_r &= ((\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j})\cdot((\cos\theta)\,\mathbf{i} + (\sin\theta)\,\mathbf{j}) \\
&= (\cos^2\theta)\,\mathbf{i}\cdot\mathbf{i} + (2\sin\theta\cos\theta)\,\mathbf{i}\cdot\mathbf{j} + (\sin^2\theta)\,\mathbf{j}\cdot\mathbf{j} \\
&= \cos^2\theta + \sin^2\theta = 1, \\
\mathbf{e}_\theta\cdot\mathbf{e}_\theta &= (-\sin\theta)^2 + (\cos\theta)^2 \\
&= \sin^2\theta + \cos^2\theta = 1, \\
\mathbf{e}_r\cdot\mathbf{e}_\theta &= (\cos\theta)(-\sin\theta) + (\sin\theta)(\cos\theta) \\
&= -\cos\theta\sin\theta + \sin\theta\cos\theta = 0.
\end{aligned}$$

Note that the result for $\mathbf{e}_\theta\cdot\mathbf{e}_r$ does not need to be explicitly stated since the dot product is commutative (see Unit 2).

Solution to Exercise 3

Since $\mathbf{A}^T\mathbf{A} = \mathbf{I}$, we have

$$\det(\mathbf{A}^T\mathbf{A}) = \det(\mathbf{A}^T)\det(\mathbf{A}) = \det(\mathbf{I}) = 1.$$

However, $\det(\mathbf{A}^T) = \det(\mathbf{A})$, so this becomes

$$(\det(\mathbf{A}))^2 = 1.$$

We can therefore deduce that $\det(\mathbf{A}) = \pm 1$.

(For rotations, $\det(\mathbf{A}) = 1$; the minus sign arises only for a reflection. We do not investigate reflections since we are concentrating on right-handed systems.)

Solution to Exercise 4

(a) In each case we compute $\mathbf{A}^T\mathbf{A}$ and compare it with the identity matrix $\mathbf{I}$.

Here, $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \mathbf{I}$.

Hence this matrix is not orthogonal.

(b) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \mathbf{I}$.

Hence this matrix is orthogonal.

(c) $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \neq \mathbf{I}$.

Hence this matrix is not orthogonal.

(d) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \mathbf{I}$.

Hence this matrix is orthogonal.

(e) $\begin{pmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} = \mathbf{I}$.

Hence this matrix is orthogonal.

Solution to Exercise 5

There are many possible choices – in fact, infinitely many. One example is

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which certainly has determinant 1. However this matrix cannot be orthogonal because when it multiplies the vector $(1 \quad 0 \quad 0)^T$, the result is $(2 \quad 1 \quad 0)^T$, a vector of different length.

Solution to Exercise 6

(a) Let the Cartesian coordinate system be in the plane of the sheet with $|\mathbf{r}| = \sqrt{x^2 + y^2}$. Then the scalar field function is

$$\Theta(x, y) = \Theta_1 + \frac{(\Theta_2 - \Theta_1)\ln(\sqrt{x^2 + y^2}/R_1)}{\ln(R_2/R_1)}.$$

The scalar field is defined over the region of the sheet. So the domain of the function is $R_2 \leq \sqrt{x^2 + y^2} \leq R_1$.

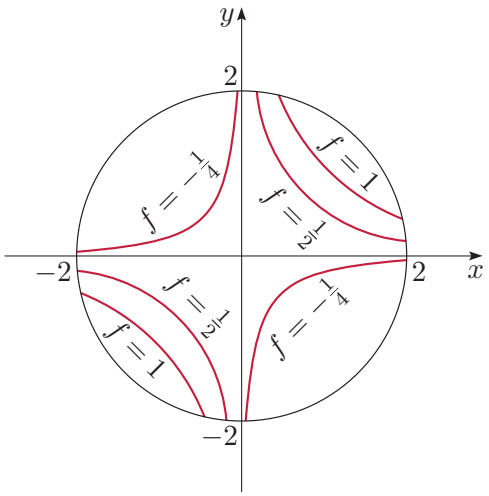
(b) In polar coordinates, $|\mathbf{r}| = r$, so the scalar field function is

$$\Theta(r, \theta) = \Theta_1 + \frac{(\Theta_2 - \Theta_1)\ln(r/R_1)}{\ln(R_2/R_1)},$$

with domain $R_2 \leq r \leq R_1$. Note that in this case, the field function is independent of θ.

Solution to Exercise 7

The domain of f is the area of the plane inside and on the circle $x^2 + y^2 = 4$ with radius 2. The contour curves are hyperbolas given by $xy = C$, where C is a constant, or equivalently, by $y = C/x$. The figure below shows the contour curves for $C = \frac{1}{2}$, $C = -\frac{1}{4}$ and $C = 1$.



Solution to Exercise 8

(a) (i) There is a hill summit inside the 60 m contour.

(ii) There might be a lake in the depression in the region of the −10 m and −20 m contours.

(b) The land is fairly level in the vicinity of the 0 m and 10 m contours, since the contour curves are widely spaced in this region.

(c) Starting at A and walking in the direction of the arrow, you would soon experience a hard climb as you ascend across the closely spaced contours leading to the top of the hill. Then there is a fairly gentle descent to the relatively level area, with a low point near the 0 m contour, and finally a gentle climb.

Solution to Exercise 9

Choose a coordinate system with the Earth's (flat) surface in the (x, y)-plane and the z-axis pointing vertically upwards. Then $\sigma(x, y, z) = Ae^{-\alpha z}$, for constants $A > 0$ and $\alpha > 0$. Knowing that $\sigma = 1.205$ at $z = 0$ tells us that $A = 1.205$. We are also told that $\sigma = 1.205/e$ when $z = 9.5 \times 10^3$. This gives $1.205/e = 1.205e^{-\alpha \times 9.5 \times 10^3}$, from which $\alpha = 1/(9.5 \times 10^3)$. Hence the scalar field is

$$\sigma(x, y, z) = 1.205e^{-z/(9.5 \times 10^3)}.$$

The domain is $z \geq 0$.

Since the field depends only on z, the contour surfaces are horizontal planes, that is, $z =$ constant.

Solution to Exercise 10

We have

$$\mathbf{F} = \frac{mgr}{R}(-\widehat{\mathbf{r}}) = -\frac{mg}{R}\mathbf{r},$$

since $r\,\widehat{\mathbf{r}} = \mathbf{r}$, so

$$\mathbf{F}(x, y, z) = \frac{-mg}{R}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}).$$

At the Earth's centre, we have $\mathbf{F}(0, 0, 0) = \mathbf{0}$.

Solution to Exercise 11

(a) The x-component of the vector field $\mathbf{J}$ is

$$J_1(x, y, z) = \mathbf{J} \cdot \mathbf{i} = \frac{Cx}{x^2 + y^2}.$$

(b) The vector $x\mathbf{i} + y\mathbf{j}$ points perpendicularly away from the z-axis. Hence the vector field lines of $\mathbf{J}$ are directed radially out from the axis of the pipe.

(c) The magnitude of $\mathbf{J}$ is given by

$$|\mathbf{J}| = \left|\frac{C(x\mathbf{i} + y\mathbf{j})}{x^2 + y^2}\right| = \frac{C}{(x^2 + y^2)^{1/2}} = \frac{C}{\rho} \quad (a \leq \rho \leq b).$$

Hence $|\mathbf{J}|$ is inversely proportional to ρ.

(d) The rate of flow outwards is the outward flow rate per unit area times the area of the surface, that is,

$$|\mathbf{J}|\,(2\pi\rho h) = (C/\rho)(2\pi\rho h) = 2\pi Ch,$$

which is independent of ρ. This means that heat energy is being conducted through the material of the pipe, from the water surface to the outside surface of the pipe, without any loss or gain of heat in the material of the pipe.

Solution to Exercise 12

Introduce a Cartesian coordinate system with the z-axis along the axis of the wire. Then the magnitude $|\mathbf{E}|$ is inversely proportional to the distance $\rho = (x^2 + y^2)^{1/2}$ from the z-axis. The direction of $\mathbf{E}$ is directly away from the wire, in the direction of the vector $x\mathbf{i} + y\mathbf{j}$. The unit vector in this direction is $\mathbf{e}_\rho = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)^{1/2}$. Hence the electric field vector at distance ρ is $\mathbf{E} = E_0\mathbf{e}_\rho/\rho$, which has magnitude E_0 when $\rho = 1$, as required. Thus in Cartesian coordinates we have the vector field

$$\mathbf{E}(x, y, z) = E_0(x\mathbf{i} + y\mathbf{j})/(x^2 + y^2) \quad (x^2 + y^2 > 0).$$

The domain statement excludes the z-axis, where the function is not defined.

The vector field lines are straight lines perpendicular to the z-axis and directed away from it.

Solution to Exercise 13

The scalar field is $f(x, y) = x^2 y$, so we have

$$\frac{\partial f}{\partial x} = 2xy, \quad \frac{\partial f}{\partial x}(-1, 2) = -4,$$
$$\frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial y}(-1, 2) = 1.$$

Thus from equation (6),

$$\mathbf{grad}\, f(-1, 2) = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = -4\mathbf{i} + \mathbf{j}.$$

The derivative in the x-direction is $\mathbf{grad}\, f \cdot \mathbf{i} = \partial f/\partial x$, which has the value -4 at $(-1, 2)$. Note that here we have $\widehat{\mathbf{d}} = \mathbf{i}$.

Solution to Exercise 14

(a) We have

$$\frac{\partial f}{\partial x} = \tfrac{1}{2}(x^2 + y^2)^{-1/2}(2x) = \frac{x}{\sqrt{x^2 + y^2}},$$
$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}.$$

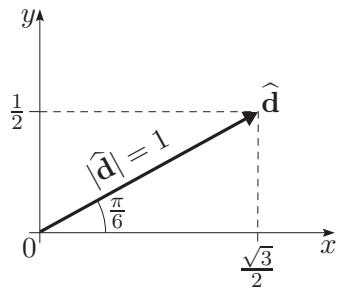
Thus from equation (6),

$$\mathbf{grad}\, f = \frac{1}{\sqrt{x^2 + y^2}}(x\mathbf{i} + y\mathbf{j})$$

and

$$\mathbf{grad}\, f(1, 1) = \frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}).$$

(b) The unit vector $\widehat{\mathbf{d}}$ is shown in the figure in the margin.



We have

$$\widehat{\mathbf{d}} = (\cos\tfrac{\pi}{6})\mathbf{i} + (\sin\tfrac{\pi}{6})\mathbf{j} = \tfrac{\sqrt{3}}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}.$$

The required derivative is

$$\begin{aligned}\mathbf{grad}\, f(1, 1) \cdot \widehat{\mathbf{d}} &= \left(\tfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})\right) \cdot \left(\tfrac{\sqrt{3}}{2}\mathbf{i} + \tfrac{1}{2}\mathbf{j}\right) \\ &= \tfrac{\sqrt{3}}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}} \\ &= \tfrac{1+\sqrt{3}}{2\sqrt{2}} \\ &\simeq 0.9659.\end{aligned}$$

Solution to Exercise 15

We have

$$\frac{\partial g}{\partial x} = 2xy - y^2, \quad \frac{\partial g}{\partial y} = x^2 - 2xy.$$

Thus from equation (6),

$$\mathbf{grad}\, g = (2xy - y^2)\mathbf{i} + (x^2 - 2xy)\mathbf{j}$$

and

$$\mathbf{grad}\, g(1,1) = \mathbf{i} - \mathbf{j}.$$

This gradient vector specifies the magnitude and direction of the steepest slope of the surface $z = g(x, y)$ at the point $(1, 1)$. Hence the magnitude of the required steepest slope is

$$|\mathbf{grad}\, g(1,1)| = |\mathbf{i} - \mathbf{j}| = \sqrt{2},$$

and the direction of steepest slope is specified by the unit vector $(\mathbf{i} - \mathbf{j})/\sqrt{2}$.

Solution to Exercise 16

The curve $x^2 - 2xy + y^2 = 9$ is a contour of $f(x, y) = x^2 - 2xy + y^2$ passing through $(0, 3)$, and $\mathbf{grad}\, f$ evaluated at $(0, 3)$ is normal to this contour. We have

$$\mathbf{grad}\, f = (2x - 2y)\mathbf{i} + (-2x + 2y)\mathbf{j}$$

and

$$\mathbf{grad}\, f(0,3) = -6\mathbf{i} + 6\mathbf{j}.$$

This, or any non-zero scalar multiple of it, is the required vector.

Solution to Exercise 17

The partial derivatives are

$$\frac{\partial \Theta}{\partial x} = \frac{-Ba}{\sqrt{x^2+y^2}} \times \frac{\frac{1}{2}(x^2+y^2)^{-1/2}}{a} \times 2x = -\frac{Bx}{x^2+y^2},$$

$$\frac{\partial \Theta}{\partial y} = -\frac{By}{x^2+y^2},$$

$$\frac{\partial \Theta}{\partial z} = 0.$$

Thus using equation (7),

$$\mathbf{grad}\, \Theta = \left(-\frac{Bx}{x^2+y^2}\right)\mathbf{i} + \left(-\frac{By}{x^2+y^2}\right)\mathbf{j}$$

and

$$\mathbf{grad}\, \Theta(-1,1,0) = \tfrac{1}{2}B\mathbf{i} - \tfrac{1}{2}B\mathbf{j} = \tfrac{1}{2}B(\mathbf{i} - \mathbf{j}).$$

Solution to Exercise 18

(a) We need the magnitude and direction of $\mathbf{grad}\, f$ at $(-1,1,1)$. From equation (7),

$$\mathbf{grad}\, f = 2xy^2z^2\mathbf{i} + 2x^2yz^2\mathbf{j} + 2x^2y^2z\mathbf{k}.$$

The maximum value of the derivative is $|\mathbf{grad}\, f|$, and it occurs in the direction of the unit vector $\mathbf{grad}\, f/|\mathbf{grad}\, f|$. At the point $(-1,1,1)$, we have

$$|\mathbf{grad}\, f| = |-2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}| = 2\sqrt{3},$$
$$\frac{\mathbf{grad}\, f}{|\mathbf{grad}\, f|} = -\tfrac{1}{\sqrt{3}}\mathbf{i} + \tfrac{1}{\sqrt{3}}\mathbf{j} + \tfrac{1}{\sqrt{3}}\mathbf{k}.$$

(b) The unit vector in the direction of $3\mathbf{i} + 4\mathbf{k}$ is given by

$$\widehat{\mathbf{d}} = (3\mathbf{i} + 4\mathbf{k})/|3\mathbf{i} + 4\mathbf{k}| = (3\mathbf{i} + 4\mathbf{k})/5 = \tfrac{3}{5}\mathbf{i} + \tfrac{4}{5}\mathbf{k}.$$

Hence the derivative of f in the direction of $3\mathbf{i} + 4\mathbf{k}$ is

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}} = \tfrac{6}{5}xy^2z^2 + \tfrac{8}{5}x^2y^2z.$$

At the point $(2,1,-1)$, this gives

$$\mathbf{grad}\, f \cdot \widehat{\mathbf{d}} = -4.$$

Solution to Exercise 19

We have

$$\frac{\partial f}{\partial x} = e^{-a(x^2+y^2)-bz} \times (-2ax) = (-2ax)\, f(x,y,z),$$
$$\frac{\partial f}{\partial y} = e^{-a(x^2+y^2)-bz} \times (-2ay) = (-2ay)\, f(x,y,z),$$
$$\frac{\partial f}{\partial z} = e^{-a(x^2+y^2)-bz} \times (-b) = (-b)\, f(x,y,z).$$

Thus from equation (7),

$$\begin{aligned}\boldsymbol{\nabla} f &= (-2ax)\, f(x,y,z)\, \mathbf{i} + (-2ay)\, f(x,y,z)\, \mathbf{j} + (-b)\, f(x,y,z)\, \mathbf{k} \\ &= -(2a(x\mathbf{i} + y\mathbf{j}) + b\mathbf{k})\, f(x,y,z),\end{aligned}$$

as required. At the origin,

$$\boldsymbol{\nabla} f(0,0,0) = -be^0\mathbf{k} = -b\mathbf{k}.$$

The x-component of $\boldsymbol{\nabla} f(1,2,3)$ is

$$\begin{aligned}\boldsymbol{\nabla} f(1,2,3) \cdot \mathbf{i} &= (-2a)\, f(1,2,3) \\ &= -2ae^{-a(1^2+2^2)-3b} \\ &= -2ae^{-5a-3b}.\end{aligned}$$

Solution to Exercise 20

We have

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\,\frac{\partial f}{\partial x} + r\cos\theta\,\frac{\partial f}{\partial y}.$$

Hence

$$\frac{\partial}{\partial \theta} = -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}.$$

Solution to Exercise 21

(a) We have $f(r,\,\theta) = r^2 - 2r\cos\theta$. Thus

$$\begin{aligned}\mathbf{grad}\, f &= \left(\mathbf{e}_r\frac{\partial}{\partial r} + \frac{1}{r}\,\mathbf{e}_\theta\frac{\partial}{\partial \theta}\right) f \\ &= \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \\ &= (2r - 2\cos\theta)\mathbf{e}_r + 2\sin\theta\,\mathbf{e}_\theta.\end{aligned}$$

(b) We have $f(r,\theta) = e^{-r\sin\theta}/r^2$. Thus

$$\frac{\partial f}{\partial r} = -\frac{e^{-r\sin\theta}\sin\theta}{r^2} - \frac{2e^{-r\sin\theta}}{r^3} = -\frac{e^{-r\sin\theta}}{r^2}\left(\sin\theta + \frac{2}{r}\right)$$

and

$$\frac{\partial f}{\partial \theta} = -\frac{\cos\theta}{r}\,e^{-r\sin\theta}.$$

Hence

$$\begin{aligned}\mathbf{grad}\, f &= \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta \\ &= -\frac{e^{-r\sin\theta}}{r^2}\left(\left(\sin\theta + \frac{2}{r}\right)\mathbf{e}_r + \cos\theta\,\mathbf{e}_\theta\right).\end{aligned}$$

Solution to Exercise 22

(a) The domain is the slab of space between (and including) the planes $z = 1$ and $z = -1$, but excluding the z-axis.

(b) At P we have $\lambda(1, 0, 0) = (\cos 0)/1 = 1$.

At Q we have $\lambda(1, \frac{\pi}{2}, 0) = (\cos 0)/1 = 1$.

At R we have $\rho = 0$, so R is not in the domain of λ.

(c) The points on the circle all have $\rho = 2$, so the required field value is $\lambda(2, \phi, \frac{1}{4}) = (\cos\frac{\pi}{8})/2 \simeq 0.4619$.

Solution to Exercise 23

We can replace $x = \rho\cos\phi$ and $y = \rho\sin\phi$ in the equation for $\mathbf{u}$, or we can write $\mathbf{u}$ as

$$\mathbf{u}(x,y,z) = \frac{e^{-z^2}(x\mathbf{i}+y\mathbf{j})}{\sqrt{x^2+y^2}\sqrt{x^2+y^2}} + \frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}\sqrt{x^2+y^2}}.$$

Then equations (14), (15) and (16) give

$$\mathbf{u}(\rho,\phi,z) = \frac{e^{-z^2}}{\rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\,\mathbf{e}_\phi.$$

Hence the cylindrical components are

$$u_\rho = \frac{e^{-z^2}}{\rho}, \quad u_\phi = \frac{1}{\rho}, \quad u_z = 0.$$

Solution to Exercise 24

(a) (i) $\rho = 2$ defines a cylindrical surface of radius 2 with its axis along the z-axis.

(ii) $\phi = \frac{\pi}{2}$ defines a half-plane perpendicular to the x-axis, containing the positive y-axis but not the z-axis.

(b) We have

$$\mathbf{e}_\rho \cdot \mathbf{e}_\phi = \left(\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right)\cdot\left(\frac{-y\mathbf{i}+x\mathbf{j}}{\sqrt{x^2+y^2}}\right) = \frac{-xy+xy}{x^2+y^2} = 0,$$

so $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ are perpendicular.

(c) The partial derivatives of F are

$$\frac{\partial F}{\partial \rho} = e^{-z}, \quad \frac{\partial F}{\partial \phi} = 0, \quad \frac{\partial F}{\partial z} = -\rho e^{-z}.$$

Hence, from equation (24), the gradient of F is

$$\boldsymbol{\nabla} F = \mathbf{grad}\, F = e^{-z}\mathbf{e}_\rho - \rho e^{-z}\mathbf{e}_z = e^{-z}(\mathbf{e}_\rho - \rho\mathbf{e}_z).$$

At the point $\rho = 5$, $\phi = 0$, $z = 0$, we have

$$\boldsymbol{\nabla} F(5,0,0) = \mathbf{e}_\rho - 5\mathbf{e}_z.$$

Solution to Exercise 25

The partial derivatives of U are

$$\frac{\partial U}{\partial x} = -\frac{2x}{(x^2+y^2)^2}, \quad \frac{\partial U}{\partial y} = -\frac{2y}{(x^2+y^2)^2}, \quad \frac{\partial U}{\partial z} = 0.$$

Hence, since $\boldsymbol{\nabla} U = \mathbf{grad}\, U$, from equation (7),

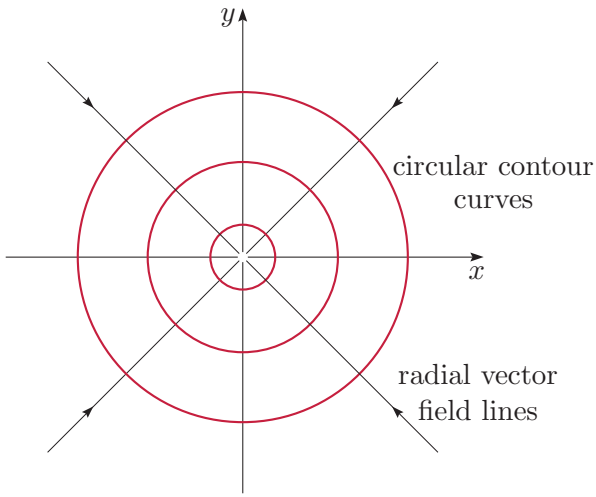
$$\boldsymbol{\nabla} U = -\frac{2}{(x^2+y^2)^2}(x\mathbf{i}+y\mathbf{j}).$$

Converting to cylindrical coordinates using equations (14), $U(\rho, \phi, z) = 1/\rho^2$. This is a cylindrically symmetric scalar field, since it depends only on ρ. So we can use equation (25) to obtain

$$\boldsymbol{\nabla} U = \mathbf{e}_\rho \frac{\partial U}{\partial \rho} = \mathbf{e}_\rho (-2)\frac{1}{\rho^3} = -\frac{2}{\rho^3}\,\mathbf{e}_\rho.$$

But from equation (15) $\mathbf{e}_\rho = (x\mathbf{i} + y\mathbf{j})/\sqrt{x^2 + y^2}$, and from equations (14) $1/\rho^3 = (x^2 + y^2)^{-3/2}$, hence the two expressions for $\boldsymbol{\nabla} U$ are identical.

The contour surfaces of U are found by putting $U = 1/\rho^2$ equal to a constant. Hence the contours are the surfaces where ρ is constant. Each such surface is cylindrical with its axis along the z-axis. We can sketch the intersections (circles) of the contours with a plane $z =$ constant (see the figure below). The vector field lines of $\boldsymbol{\nabla} U$ are directed perpendicularly inwards towards the z-axis. (The vector field lines are not continuous through the z-axis, which is not in the domain of U.)



Solution to Exercise 26

The partial derivatives are

$$\frac{\partial f}{\partial \rho} = -\frac{\cos z}{\rho^2}, \quad \frac{\partial f}{\partial \phi} = 0, \quad \frac{\partial f}{\partial z} = -\frac{\sin z}{\rho}.$$

Hence, using equation (24),

$$\mathbf{grad}\, f = \mathbf{e}_\rho \left(-\frac{\cos z}{\rho^2}\right) + \mathbf{e}_z \left(-\frac{\sin z}{\rho}\right).$$

So the components $(\mathbf{grad}\, f)_\rho$, $(\mathbf{grad}\, f)_\phi$ and $(\mathbf{grad}\, f)_z$ are $(-\cos z)/\rho^2$, 0 and $(-\sin z)/\rho$, respectively.

Solution to Exercise 27

(a) Here $r = \sqrt{5^2 + 0^2 + 0^2} = 5$. The point is in the (x, y)-plane, so $\theta = \frac{\pi}{2}$. The point is on the x-axis, so $\phi = 0$. Hence the spherical coordinates are $(5, \frac{\pi}{2}, 0)$.

(b) The point is on the positive y-axis 5 units from the origin, so the spherical coordinates are $(5, \frac{\pi}{2}, \frac{\pi}{2})$.

Solution to Exercise 28

From equations (26), (27), (29) and (32), we have

$$\begin{aligned}\mathbf{F}(r, \theta, \phi) &= \frac{-r \sin\theta \sin\phi\, \mathbf{i} + r \sin\theta \cos\phi\, \mathbf{j}}{r^2} \\ &= \frac{\sin\theta}{r}\,(-\sin\phi\, \mathbf{i} + \cos\phi\, \mathbf{j}) \\ &= \frac{\sin\theta}{r}\,\mathbf{e}_\phi,\end{aligned}$$

so using equation (35), the spherical components of $\mathbf{F}$ are

$$F_r = F_\theta = 0, \quad F_\phi = \frac{\sin\theta}{r}.$$

Solution to Exercise 29

Using equation (35), the spherical components are

$$(\mathbf{grad}\, V)_r = I_0/r^2, \quad (\mathbf{grad}\, V)_\theta = (\mathbf{grad}\, V)_\phi = 0.$$

Solution to Exercise 30

(a) $\theta = \frac{\pi}{6}$ defines a cone with its axis along the positive z-axis, its apex at the origin, and half-angle $\frac{\pi}{6}$.

(b) We have

$$\begin{aligned}|\mathbf{e}_r|^2 &= \mathbf{e}_r \cdot \mathbf{e}_r \\ &= (\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta) \\ &\quad \cdot (\mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta) \\ &= (\sin\theta\cos\phi)^2 + (\sin\theta\sin\phi)^2 + \cos^2\theta \\ &= \sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta \\ &= \sin^2\theta + \cos^2\theta = 1.\end{aligned}$$

(c) In spherical coordinates, $U = r$ and the only non-zero partial derivative is $\partial U/\partial r = 1$. Hence $\mathbf{grad}\, U = (1)\mathbf{e}_r = \mathbf{e}_r$. The point $(5, \frac{\pi}{2}, \pi)$ is on the negative x-axis, so $\mathbf{e}_r = -\mathbf{i}$ at this point.

(Alternatively, using equation (30),

$$\begin{aligned}\mathbf{e}_r &= \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta \\ &= \mathbf{i}(-1) + \mathbf{0} + \mathbf{0} = -\mathbf{i}.)\end{aligned}$$

Solution to Exercise 31

(a) At A, $r = 3$, $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{2}$ and $V(3, \frac{\pi}{2}, \frac{\pi}{2}) = -M/3^3 = -M/27$.

At B, $r = 1$, $\theta = 0$, $\phi = 0$ and $V(1, 0, 0) = 2M$. (B is on the z-axis, where $\phi = 0$ by convention.)

At C, $r = 1$, $\theta = \pi$, $\phi = 0$ and $V(1, \pi, 0) = 2M$.

(b) To find $\mathbf{grad}\, V$, we first determine the partial derivatives

$$\frac{\partial V}{\partial r} = -\frac{3M(3\cos^2\theta - 1)}{r^4}, \quad \frac{\partial V}{\partial \theta} = \frac{M(6\cos\theta(-\sin\theta))}{r^3}, \quad \frac{\partial V}{\partial \phi} = 0.$$

Evaluating these at A yields

$$\frac{\partial V}{\partial r}(3, \tfrac{\pi}{2}, \tfrac{\pi}{2}) = M/27, \quad \frac{\partial V}{\partial \theta}(3, \tfrac{\pi}{2}, \tfrac{\pi}{2}) = 0.$$

Hence, at A, $\mathbf{grad}\, V = \mathbf{e}_r\, M/27$. The magnitude of the force $\mathbf{F}$ is $|-\mathbf{grad}\, V| = M/27$, and the direction is given by the direction of $-\mathbf{e}_r$ at A, that is, the negative y-direction.

Unit 16

Further vector calculus

Introduction

In Unit 15 you saw that the spatial variations of a scalar field can be described by the gradient vector field, which can be expressed in terms of the three partial derivatives of the scalar field. Vector fields are more complicated than scalar fields. For a given Cartesian coordinate system there are nine partial derivatives associated with a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ at each field point. In the first two sections of this unit we will introduce two new fields that are combinations of these partial derivatives and that represent important physical properties of the vector field $\mathbf{F}$. One of these fields, a scalar field called the *divergence* of $\mathbf{F}$, is the subject of Section 1. The other, a vector field called the *curl* of $\mathbf{F}$, is the subject of Section 2.

These partial derivatives are $\partial F_1/\partial x$, $\partial F_1/\partial y$, $\partial F_1/\partial z$, $\partial F_2/\partial x$, $\partial F_2/\partial y$, $\partial F_2/\partial z$, $\partial F_3/\partial x$, $\partial F_3/\partial y$, $\partial F_3/\partial z$.

In Section 3 we measure the work done by a force in moving a particle from one point to another in a vector field, by introducing the scalar line integral. Finally, Section 4 develops links between gradient, curl and the scalar line integral.

This unit will also build on the concepts of kinetic energy and potential energy, introduced in Unit 9.

1 Divergence of a vector field

We will approach divergence in two different, but equivalent, ways. In Subsections 1.1 and 1.2 we define divergence mathematically in terms of partial derivatives, using the vector differential operator $\boldsymbol{\nabla}$, and give you practice in calculating the divergence of given vector fields. Subsection 1.3 discusses the problem of modelling the flow of heat energy in a uranium fuel rod, and predicting the temperature field in the rod. This enables us to develop a physical interpretation of divergence as the rate of heat outflow per unit volume at each point in the rod – one of many examples of divergence in the physical world.

1.1 Defining divergence

The gradient $\mathbf{grad}\, f$ of a scalar field f can be expressed as $\boldsymbol{\nabla} f$, where $\boldsymbol{\nabla}$ (del) is the vector differential operator

You saw the del operator, denoted $\boldsymbol{\nabla}$, in Unit 15, Subsection 3.4.

$$\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

We now form the dot product of $\boldsymbol{\nabla}$ with an arbitrary vector field $\mathbf{F}(x, y, z)$, treating $\boldsymbol{\nabla}$ as if it is a vector.

We write the order of symbols in the dot product as $\nabla \cdot \mathbf{F}$, so

$$\nabla \cdot \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}),$$

The order is important here. We do *not* write $\mathbf{F} \cdot \nabla$, since this would result in a differential operator looking for something to operate on.

where we have written the vector field $\mathbf{F}$ in terms of its Cartesian component fields $F_1(x, y, z)$, etc. We can evaluate this expression by multiplying the brackets, letting the differential operators act on the scalar component functions, and using the rule for the dot products of the unit vectors. For example, multiplying the first term in each bracket yields $(\partial F_1/\partial x)\,\mathbf{i} \cdot \mathbf{i} = \partial F_1/\partial x$. Terms with dot products of mutually perpendicular unit vectors evaluate to zero, so we are left with the sum of the three terms $\partial F_1/\partial x$, $\partial F_2/\partial y$ and $\partial F_3/\partial z$. This combination of partial derivatives is defined as the *divergence* of the vector field $\mathbf{F}$, or $\operatorname{div}\mathbf{F}$ for short. You can see that $\operatorname{div}\mathbf{F}$ is a scalar quantity that can be evaluated at each point in the domain of $\mathbf{F}$. Thus $\operatorname{div}\mathbf{F}$ is a scalar field.

Divergence

The **divergence** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ in **Cartesian coordinates** is a scalar field given by

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}. \tag{1}$$

$\operatorname{div}\mathbf{F}$ and $\nabla \cdot \mathbf{F}$ are alternative notations for the divergence of $\mathbf{F}$. Note that we do not embolden 'div' in $\operatorname{div}\mathbf{F}$ since $\operatorname{div}\mathbf{F}$ is a scalar.

A physical interpretation of divergence will be given in Subsection 1.3. Meanwhile we show how the divergence of a given vector field can be calculated at specified points.

1.2 Calculating divergence

To calculate the divergence of a given vector field $\mathbf{F}$, we need to identify its components F_1, F_2 and F_3, work out the partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$ and $\partial F_3/\partial z$, and add them as in equation (1).

Example 1

(a) Determine the divergence of each of the following vector fields:

$$\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k},$$
$$\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$$
$$\mathbf{V}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 5\mathbf{k}.$$

(b) Evaluate $\operatorname{div}\mathbf{F}$ at the origin and at the point $(1, 2, 3)$.

Solution

(a) The components of $\mathbf{F}$ are $F_1 = x^2$, $F_2 = y^2$, $F_3 = z^2$. Hence

$$\operatorname{div}\mathbf{F} = 2x + 2y + 2z = 2(x + y + z).$$

Similarly, $r_1 = x$, $r_2 = y$, $r_3 = z$. Hence

$$\operatorname{div}\mathbf{r} = 1 + 1 + 1 = 3.$$

Also, $V_1 = -y$, $V_2 = x$, $V_3 = 5$. Hence

$$\operatorname{div}\mathbf{V} = 0 + 0 + 0 = 0.$$

(b) The value of $\operatorname{div}\mathbf{F}$ at $(0, 0, 0)$ is 0, and its value at $(1, 2, 3)$ is $2(1 + 2 + 3) = 12$.

Exercise 1

Determine the scalar field $\operatorname{div}\mathbf{F}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} - yz\mathbf{k}$, and evaluate $\operatorname{div}\mathbf{F}$ at the point $(3, -1, 2)$.

Exercise 2

Find $\boldsymbol{\nabla} \cdot \mathbf{F}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$, and hence evaluate $\boldsymbol{\nabla} \cdot \mathbf{F}(1, 2, 3)$.

Exercise 3

Consider the three-dimensional vector field $\mathbf{F} = \widehat{\mathbf{r}}/|\mathbf{r}|^n$ $(\mathbf{r} \neq \mathbf{0})$, where $\mathbf{r}$ is a position vector, $\widehat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$, and n is a positive integer.

Express the field in Cartesian form, and show that $\mathbf{F}$ has zero divergence everywhere only when $n = 2$.

When vector fields are given in cylindrical or spherical coordinates, it is not always convenient to use the Cartesian expression for divergence. The cylindrical and spherical coordinate expressions for divergence are quite complicated, and we will derive only the expression for cylindrical coordinates.

First, recall that the gradient vector operator in cylindrical coordinates is given by

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}.$$

See Unit 15, Subsection 4.1.

Recall that in cylindrical coordinates, we write

$$\mathbf{F} = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z.$$

To compute the divergence $\boldsymbol{\nabla} \cdot \mathbf{F}$, we must therefore calculate

$$\left(\mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z} \right) \cdot \left(F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z \right).$$

Note that $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ do not depend on ρ or z.

Now $\mathbf{e}_\rho$ and $\mathbf{e}_\phi$ depend on ϕ. Indeed, from the relationships

$$\begin{pmatrix}\mathbf{e}_\rho\\ \mathbf{e}_\phi\\ \mathbf{e}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{i}\\ \mathbf{j}\\ \mathbf{k}\end{pmatrix},$$

we see, for example, that

$$\mathbf{e}_\rho = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}.$$

Therefore

$$\frac{\partial \mathbf{e}_\rho}{\partial \phi} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j} = \mathbf{e}_\phi.$$

Exercise 4

Find a similar expression for $\dfrac{\partial \mathbf{e}_\phi}{\partial \phi}$.

Now, returning to the calculation of $\boldsymbol{\nabla}\cdot\mathbf{F}$, we proceed term by term as follows:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\mathbf{F} &= \left(\mathbf{e}_\rho\frac{\partial}{\partial\rho} + \mathbf{e}_\phi\frac{1}{\rho}\frac{\partial}{\partial\phi} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &= \mathbf{e}_\rho\cdot\frac{\partial}{\partial\rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) + \frac{1}{\rho}\mathbf{e}_\phi\cdot\frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &\quad + \mathbf{e}_z\cdot\frac{\partial}{\partial z}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z).\end{aligned}$$

Since none of $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ or $\mathbf{e}_z$ depends on ρ, we have

$$\frac{\partial}{\partial\rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{\partial F_\rho}{\partial\rho}\mathbf{e}_\rho + \frac{\partial F_\phi}{\partial\rho}\mathbf{e}_\phi + \frac{\partial F_z}{\partial\rho}\mathbf{e}_z,$$

$\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ form a right-handed set of unit vectors, thus are mutually perpendicular.

so

$$\mathbf{e}_\rho\cdot\frac{\partial}{\partial\rho}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{\partial F_\rho}{\partial\rho}.$$

Next, using the results $\partial\mathbf{e}_\rho/\partial\phi = \mathbf{e}_\phi$, $\partial\mathbf{e}_\phi/\partial\phi = -\mathbf{e}_\rho$ and $\partial\mathbf{e}_z/\partial\phi = \mathbf{0}$, we have

$$\begin{aligned}&\frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z)\\ &= \frac{\partial F_\rho}{\partial\phi}\mathbf{e}_\rho + F_\rho\frac{\partial\mathbf{e}_\rho}{\partial\phi} + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi + F_\phi\frac{\partial\mathbf{e}_\phi}{\partial\phi} + \frac{\partial F_z}{\partial\phi}\mathbf{e}_z\\ &= \frac{\partial F_\rho}{\partial\phi}\mathbf{e}_\rho + F_\rho\mathbf{e}_\phi + \frac{\partial F_\phi}{\partial\phi}\mathbf{e}_\phi - F_\phi\mathbf{e}_\rho + \frac{\partial F_z}{\partial\phi}\mathbf{e}_z.\end{aligned}$$

So

$$\frac{1}{\rho}\mathbf{e}_\phi\cdot\frac{\partial}{\partial\phi}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi}.$$

Finally, since none of $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ depends on z, the contribution from the z-component of $\boldsymbol{\nabla}$ is

$$\mathbf{e}_z\cdot\frac{\partial}{\partial z}(F_\rho\mathbf{e}_\rho + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z) = \frac{\partial F_z}{\partial z}.$$

So we have finally arrived at the formula for $\operatorname{div}\mathbf{F}$ in cylindrical coordinates:

$$\operatorname{div}\mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}.$$

Divergence in cylindrical and spherical coordinates

The **divergence** of a vector field $\mathbf{F}(\rho, \theta, z) = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z$ is given in **cylindrical coordinates** by

$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F} = \frac{\partial F_\rho}{\partial \rho} + \frac{1}{\rho}F_\rho + \frac{1}{\rho}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}. \qquad (2)$$

For $\mathbf{F}(r, \theta, \phi) = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$ in **spherical coordinates**, the divergence is given by

$$\operatorname{div}\mathbf{F} = \frac{\partial F_r}{\partial r} + \frac{1}{r}\left(\frac{\partial F_\theta}{\partial \theta} + 2F_r\right) + \frac{1}{r\sin\theta}\left(\frac{\partial F_\phi}{\partial \phi} + F_\theta \cos\theta\right). \qquad (3)$$

This follows from a similar but more complicated derivation that we do not consider here.

Example 2

Consider the vector field $\mathbf{F} = \widehat{\mathbf{r}}/|\mathbf{r}|^n$ $(\mathbf{r} \neq \mathbf{0})$, where $\mathbf{r}$ is a position vector, $\widehat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$, and n is a positive integer.

This is the same field as in Exercise 3.

Express $\mathbf{F}$ in spherical coordinates, and hence confirm that $\operatorname{div}\mathbf{F} = 0$ everywhere only when $n = 2$.

Solution

The spherical form of the field is $\mathbf{F}(r, \theta, \phi) = \mathbf{e}_r/r^n$ $(r > 0)$, so the field has components $F_r = 1/r^n$ and $F_\theta = F_\phi = 0$. We can use the spherical expression for $\operatorname{div}\mathbf{F}$ given in equation (3) to obtain

Recall that $r = |\mathbf{r}|$ and $\mathbf{e}_r = \widehat{\mathbf{r}}$.

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{\partial(1/r^n)}{\partial r} + \frac{2}{r}(1/r^n) \\ &= (-n)\frac{1}{r^{n+1}} + \frac{2}{r^{n+1}} \\ &= \frac{2-n}{r^{n+1}} \quad (r > 0),\end{aligned}$$

which is zero only when $n = 2$.

You can see the advantage of using spherical coordinates in Example 2. (Compare the solution with that for Exercise 3.)

Exercise 5

Express the position vector $\mathbf{r}$ in spherical coordinates, and hence determine $\operatorname{div}\mathbf{r}$ using the spherical expression for divergence as given by equation (3).

1.3 Divergence in physical laws

Our definition of div $\mathbf{F}$ as a sum of partial derivatives gives little clue to the physical meaning of divergence. In this subsection we describe one physical meaning of divergence by considering the flow of heat energy in a uranium fuel rod. In the steady state, the rate of flow of heat energy out of any small region of the rod must be equal to the rate at which heat is generated by nuclear fission in that small region. We will show that this heat–energy balance can be expressed by the equation

Our model assumes that there are no other heat losses or gains.

$$\operatorname{div}\mathbf{J} = S,$$

where div $\mathbf{J}$ represents the *rate of heat outflow per unit volume* at a point in the rod, and S is the *rate at which heat is generated per unit volume at that point*. (The scalar field S is sometimes referred to as the *heat source density*.) This equation represents the *law of energy conservation* for heat flow in the rod.

See Unit 15, Subsection 2.3.

You saw in Unit 15 that the heat flow rate in a heat conductor may be represented as a vector field, $\mathbf{J}$, where the direction is that of the heat flow, and the magnitude is given by

$$|\mathbf{J}| = \lim_{A\to 0} \frac{\text{heat flow rate across surface area } A}{A}.$$

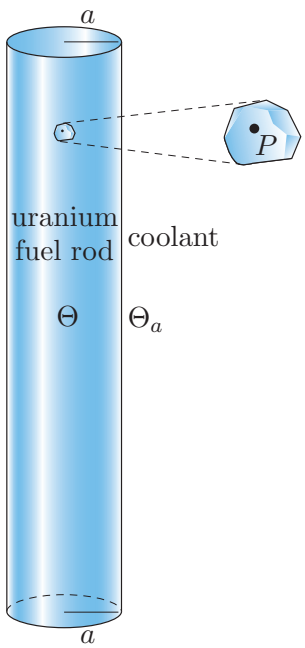


Figure 1 Uranium fuel rod in a nuclear reactor

For a uranium fuel rod in a nuclear reactor (see Figure 1), heat is generated throughout the body of the rod by nuclear fission of uranium atoms. The heat generated in the rod flows by conduction through the rod and out into the surrounding coolant, a flowing gas or liquid that takes the heat away to drive turbines. In the steady state, the coolant establishes a uniform, relatively cool, constant temperature Θ_a on the outside surface of the rod (where the radius of the rod is a). An important question in reactor design is: how does the temperature distribution Θ depend on the heat source and on the constant surface temperature Θ_a established by the coolant? In preparation for this, we now derive the equation $\operatorname{div}\mathbf{J} = S$.

When the reactor has been in operation for some time, a steady state is reached where the temperature field in the rod stays constant in time and the net outflow of heat from any region of the rod is balanced by the total heat production by nuclear fission in that region. Let us consider this energy balance more closely. If P is a point in a small region of volume V inside the rod, and q is the rate at which heat is generated by nuclear fission in the region, then the quantity

$$S = \lim_{V\to 0} \frac{q}{V}$$

is the rate (in $\mathrm{W\,m^{-3}}$) at which heat is generated per unit volume at P. The rate S of heat generation may vary from point to point in the rod, and is a scalar field.

In Figure 2, the point P has position vector $\mathbf{r}$ and lies inside a small region R, with volume V, entirely within the rod. If the region R is small enough, then the field S is effectively constant throughout the small region and is approximately equal to $S(\mathbf{r})$. So the net rate at which heat is generated in R is approximately $S(\mathbf{r})\,V$ (in watts). In the steady state, this rate of heat generation is balanced by the net rate of outflow of heat from the region R into neighbouring regions of the rod. Thus we have

$$\text{net outflow rate from small region containing } P \simeq S(\mathbf{r})\,V.$$

Dividing both sides by V, we obtain

$$\frac{\text{net outflow rate from small region containing } P}{V} \simeq S(\mathbf{r}). \tag{4}$$

Let us take the limit as the region R becomes smaller and eventually shrinks onto the point P, that is, as $V \to 0$. Then while the right-hand side of approximation (4) remains constant, the left-hand side becomes the *net outflow rate per unit volume at* P, and the approximation becomes an equality. Thus we have

$$\text{net outflow rate per unit volume at } P = S(\mathbf{r}). \tag{5}$$

We can evaluate the net outflow rate per unit volume on the left-hand side of this equation by considering a region R in the shape of a small cube centred on P, as shown in Figure 3. We will determine the net rate of outflow across the three pairs of parallel faces of the cube, and then take the limit as $V \to 0$ to give the net outflow rate per unit volume. Consider first the two parallel faces normal to the x-direction and separated from one another by a distance L, the side length of the cube. Let the left-hand face be in the plane $x = X$, with its midpoint at (X, Y, Z). If $\mathbf{J} = J_1\mathbf{i} + J_2\mathbf{j} + J_3\mathbf{k}$ is the heat flow vector field (in $\mathrm{W\,m^{-2}}$), only the component of $\mathbf{J}$ normal to this face, in the direction of negative x, will contribute to the outflow rate across this face. (The y- and z-components of $\mathbf{J}$ represent flow in the plane of the face.) If the cube is small enough, we can ignore any variation of $\mathbf{J}$ on the face and take its value $\mathbf{J}(X, Y, Z)$ at the centre of the face. The unit vector in the negative x-direction is $-\mathbf{i}$, so the outflow rate per unit area (in $\mathrm{W\,m^{-2}}$) across this face is $-\mathbf{J}\cdot\mathbf{i} = -J_1$, and the total outflow rate (in watts) across the area L^2 of the face is approximately $-J_1(X, Y, Z)\,L^2$. Similarly, the total outflow rate across the parallel face in the plane $x = X + L$ is $J_1(X + L, Y, Z)\,L^2$, since only the component of $\mathbf{J}$ in the positive x-direction contributes. The net outflow rate across the two faces is therefore approximately

$$[-J_1(X, Y, Z) + J_1(X + L, Y, Z)]\,L^2.$$

(In Figure 3, the Cartesian coordinate system is oriented so that the coordinate planes are parallel to the plane faces of the cube. We use an orientation of x, y and z that is different from that used generally in this unit.)

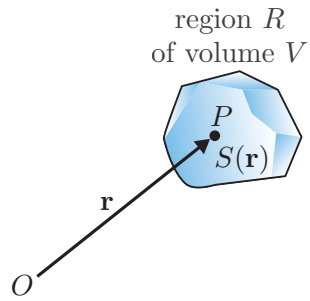


Figure 2 Small region containing point P inside the fuel rod

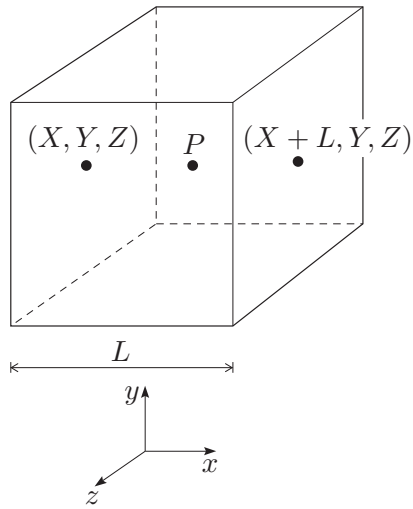


Figure 3 Cube of side length L centred at point P

The units of outflow rate per unit volume are therefore $\mathrm{W\,m^{-3}}$.

Dividing by the volume $V = L^3$ and taking the limit as $L \to 0$ (equivalent to $V \to 0$) gives the contribution from these two parallel faces to the net rate of outflow per unit volume as

$$\lim_{L\to 0} \frac{J_1(X+L,Y,Z) - J_1(X,Y,Z)}{L}.$$

You should recognise this limit from Unit 7. It is the partial derivative of J_1 with respect to x at the point (X,Y,Z). In other words, it is the partial derivative $\partial J_1(X,Y,Z)/\partial x$. In the same way, you can see that the contributions to the net rate of outflow per unit volume from the other two pairs of parallel faces of the cube are $\partial J_2(X,Y,Z)/\partial y$ and $\partial J_3(X,Y,Z)/\partial z$. The net rate of heat outflow per unit volume at the point P is the sum of these three partial derivatives, namely,

Taking the limit as $L \to 0$ in all three directions places (X,Y,Z) at the point P.

$$\frac{\partial J_1}{\partial x}(X,Y,Z) + \frac{\partial J_2}{\partial y}(X,Y,Z) + \frac{\partial J_3}{\partial z}(X,Y,Z).$$

You will recognise this expression as the *divergence* of $\mathbf{J}$.

Hence at each point P in the rod, we have

$$\text{net outflow rate per unit volume} = \operatorname{div}\mathbf{J},$$

so from equation (5), energy balance requires

$$\operatorname{div}\mathbf{J} = S \tag{6}$$

Note that this equation is dimensionally correct.

to be satisfied at each point P in the rod.

The equation $\operatorname{div}\mathbf{J} = S$ is a first-order partial differential equation for the vector field $\mathbf{J} = \mathbf{J}(x,y,z)$. It may be written as

$$\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} = S, \tag{7}$$

where J_1, J_2, J_3 and S are functions of x, y and z.

You have seen (in Unit 15, Subsection 3.3) that we can describe the heat flow vector $\mathbf{J}$ by Fourier's law as

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta, \tag{8}$$

where Θ is the temperature field and κ is the thermal conductivity. If κ is constant throughout the fuel rod, then the divergence of equation (8), divided by $-\kappa$, can be written as

$$-\frac{1}{\kappa}\operatorname{div}\mathbf{J} = \operatorname{div}(\mathbf{grad}\,\Theta) = \frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2},$$

which, from equation (6), gives us the second-order partial differential equation

$$\frac{\partial^2\Theta}{\partial x^2} + \frac{\partial^2\Theta}{\partial y^2} + \frac{\partial^2\Theta}{\partial z^2} = -\frac{S}{\kappa}. \tag{9}$$

This equation for the temperature distribution Θ is called *Poisson's equation*.

Knowing S and κ, and the appropriate boundary conditions, equation (9) can be solved for the temperature field Θ.

Example 3

Steady-state conductive heat flow is governed by the partial differential equation

$$\operatorname{div}\mathbf{J} = S,$$

where $\mathbf{J}$ (in $\mathrm{W\,m^{-2}}$) is the heat flow per unit area, and the scalar field S (in $\mathrm{W\,m^{-3}}$) is the heat generated per unit volume. Nuclear fission of uranium atoms in a long cylindrical fuel rod of radius 0.02 m generates heat at a constant rate S uniformly throughout the rod. When the axis of the rod is along the z-axis, the solution of the equation describing steady-state heat flow in the rod is known to be of the form

$$\mathbf{J}(x, y, z) = A(x\mathbf{i} + y\mathbf{j}),$$

where A is a constant.

The description of the rod as 'long' implies that we can ignore any anomalies in the heat flow or heat generation at the ends of the rod, i.e. we ignore any 'end effects'.

(a) Determine A when $S = 4 \times 10^6$.

(b) Hence determine the magnitude and direction of the vector $\mathbf{J}$ at:

(i) points on the z-axis

(ii) a point T with Cartesian coordinates $(0.005, 0, 0)$

(iii) a point Q with Cartesian coordinates $(0.015, 0, 0)$.

Solution

(a) We find that

$$\operatorname{div}\mathbf{J} = \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} = \frac{\partial (Ax)}{\partial x} + \frac{\partial (Ay)}{\partial y} = A + A = 2A.$$

Now from equation (6), we have $2A = S$, where S is given as $4 \times 10^6\,\mathrm{W\,m^{-3}}$, so $A = 2 \times 10^6$.

(b) (i) On the z-axis, $x = y = 0$, so $\mathbf{J} = \mathbf{0}$. The magnitude is zero and there is no direction associated with the zero vector.

(ii) At point T, $\mathbf{J}(0.005, 0, 0) = 2 \times 10^6 \times (0.005\mathbf{i}) = 10^4\mathbf{i}$, which is a vector pointing in the positive x-direction, that is, radially outwards, of magnitude $10^4\,\mathrm{W\,m^{-2}} = 10\,\mathrm{kW\,m^{-2}}$.

(iii) Similarly, at Q, $\mathbf{J} = 2 \times 10^6 \times (0.015\mathbf{i})$, which is a vector directed in the positive x-direction of magnitude $3 \times 10^4\,\mathrm{W\,m^{-2}} = 30\,\mathrm{kW\,m^{-2}}$.

You will be asked to confirm this expression for **J** in cylindrical coordinates in Exercise 7.

Figure 4 depicts the heat flow vector **J** at the points T and Q described in Example 3. The vector field $\mathbf{J}(x, y, z)$ can be expressed in cylindrical coordinates as $\mathbf{J}(\rho, \phi, z) = A\rho\,\mathbf{e}_\rho$, so **J** points radially outwards from and perpendicular to the z-axis everywhere, and $|\mathbf{J}|$ is the same at all points on a cylindrical surface of radius ρ.

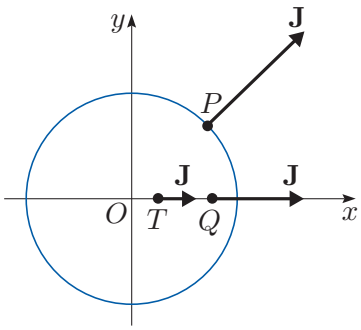


Figure 4 Cross-section of the fuel rod used in Example 3 and Exercise 6

Exercise 6

This exercise extends Example 3. Determine the magnitude of the heat flow field $\mathbf{J} = A(x\mathbf{i} + y\mathbf{j})$ at any point P on the outer cylindrical surface of the rod (see Figure 4). Hence deduce the total rate of heat flow out of a one-metre length of the rod. Determine also the total rate of heat generated in the one-metre length.

Exercise 7

Express the vector field $\mathbf{J}(x, y, z) = A(x\mathbf{i} + y\mathbf{j})$, given in Example 3, in cylindrical coordinates, and hence determine $\operatorname{div}\mathbf{J}$ using the cylindrical expression for divergence (equation (2)).

Now let us return to the general problem of the fuel rod. Recall that we had arrived at the two equations

$$\operatorname{div}\mathbf{J} = S, \tag{10}$$

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta. \tag{11}$$

The shape of the fuel rod makes it particularly appropriate to use cylindrical coordinates, and armed with the formula for div in cylindrical coordinates, we can make rapid progress. We take the z-axis along the centre of the rod so that the symmetry of the situation allows us to make further simplifications. The flow rate vector $\mathbf{J} = J_\rho\mathbf{e}_\rho + J_\phi\mathbf{e}_\phi + J_z\mathbf{e}_z$ cannot depend on the angle ϕ, because we have assumed that the material of the rod is uniform. Hence $J_\phi = 0$. Similarly, if we assume that the rod is very long in comparison with its thickness, then we can neglect the effects of the ends of the rod. In these circumstances **J** will not depend on z either, and $J_z = 0$. Thus the formula for $\operatorname{div}\mathbf{J}$ in cylindrical coordinates reduces to

$$\operatorname{div}\mathbf{J} = \frac{\partial J_\rho}{\partial\rho} + \frac{1}{\rho}J_\rho,$$

and equation (10) becomes

$$\frac{\partial J_\rho}{\partial\rho} + \frac{1}{\rho}J_\rho = S. \tag{12}$$

Now equation (11) gives

$$\mathbf{J} = -\kappa\,\mathbf{grad}\,\Theta = -\kappa\left(\mathbf{e}_\rho\frac{\partial\Theta}{\partial\rho} + \frac{1}{\rho}\mathbf{e}_\phi\frac{\partial\Theta}{\partial\phi} + \mathbf{e}_z\frac{\partial\Theta}{\partial z}\right),$$

so $J_\rho = -\kappa\,\partial\Theta/\partial\rho$. Since κ is constant, equation (12) leads to the following second-order differential equation for Θ:

$$\frac{\partial^2\Theta}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Theta}{\partial\rho} = -\frac{S}{\kappa}.$$

In fact, because Θ does not depend on ϕ or z, this partial differential equation can be written as a second-order *ordinary* differential equation:

$$\frac{d^2\Theta}{d\rho^2} + \frac{1}{\rho}\frac{d\Theta}{d\rho} = -\frac{S}{\kappa}.$$

However, it is not a constant-coefficient equation, so the methods of Unit 1 are not directly applicable. Fortunately, we can reduce the equation to a pair of first-order equations, each of which can be solved. Replacing $d\Theta/d\rho$ by u, we have

$$\frac{du}{d\rho} + \frac{1}{\rho}u = -\frac{S}{\kappa}, \quad \frac{d\Theta}{d\rho} = u.$$

The first of these is a linear differential equation, with integrating factor

$$\exp\left(\int \frac{1}{\rho}\,d\rho\right) = \exp(\ln\rho) = \rho.$$

Multiplying through by this factor gives

$$\frac{d}{d\rho}(u\rho) = -\frac{S}{\kappa}\rho,$$

so

$$u\rho = -\tfrac{1}{2}\frac{S}{\kappa}\rho^2 + C,$$

where C is an arbitrary constant. If we assume that Θ has a maximum at the centre of the rod (which seems eminently reasonable!), then we must have

$$\frac{d\Theta}{d\rho} = u = 0 \quad \text{when} \quad \rho = 0,$$

so $C = 0$. Thus

$$u = \frac{d\Theta}{d\rho} = -\tfrac{1}{2}\frac{S}{\kappa}\rho,$$

and

$$\Theta = -\tfrac{1}{4}\frac{S}{\kappa}\rho^2 + D,$$

where D is another arbitrary constant. Finally, since the temperature on the edge of the rod (when $\rho = a$) is known to be Θ_a, we have

$$\Theta_a = \Theta(a) = -\tfrac{1}{4}\frac{S}{\kappa}a^2 + D,$$

thus

$$D = \Theta_a + \tfrac{1}{4}\frac{S}{\kappa}a^2.$$

Therefore

$$\Theta = \Theta_a + \tfrac{1}{4}\frac{S}{\kappa}a^2 - \tfrac{1}{4}\frac{S}{\kappa}\rho^2 = \Theta_a + \frac{S}{4\kappa}(a^2 - \rho^2)$$

is the solution to the problem.

Exercise 8

(a) Suppose that a laser beam transports energy through air as a cylindrical beam of light, that is, in a uniform parallel beam, at the rate of $5000\,\mathrm{W\,m^{-2}}$. Taking the z-axis to be in the beam direction, along the axis of the cylinder, we can write the energy flow field as $\mathbf{N}(x, y, z) = 5000\mathbf{k}$, where $\mathbf{k}$ is a unit vector in the direction of the positive z-axis. Determine $\operatorname{div}\mathbf{N}$.

(b) When the laser beam shines through fog, it becomes weaker and its intensity falls off exponentially with distance. If the laser is placed so that the beam originates at $z = 0$, then the energy flow is given by $\mathbf{N}(x, y, z) = 5000e^{-\alpha z}\mathbf{k}$, where α is a positive constant (the absorption coefficient of the fog). Determine $\operatorname{div}\mathbf{N}$.

You have seen that for steady-state heat flow in a uranium rod with heat flow field $\mathbf{J}$, $\operatorname{div}\mathbf{J}$ is equal to the heat source density at each point. In fact, for *any* steady-state heat flow field $\mathbf{J}$, $\operatorname{div}\mathbf{J}$ is equal to the heat source density at each point. A similar result applies to the laser beam in Exercise 8, where $\operatorname{div}\mathbf{N}$ is equal to the source density of light energy in the beam. In Exercise 8(a) you found $\operatorname{div}\mathbf{N} = 0$. This result represents the fact that there are no light sources or sinks in the air through which the laser beam passes. On the other hand, when the beam passes through fog, which absorbs light and so attenuates the beam (hence the factor $e^{-\alpha z}$), the fog acts as a sink of light. The presence of this sink of light is represented by the fact that $\operatorname{div}\mathbf{N}$ is negative.

A vector field $\mathbf{F}$ for which $\operatorname{div}\mathbf{F} = 0$ everywhere is called **solenoidal**.

When a source density S is negative, the source is commonly called a *sink* (cf. the terminology for stability in Unit 12).

Exercise 9

Consider the vector field

$$\mathbf{f}(\rho, \phi, z) = \frac{\mathbf{e}_\rho}{\rho^n},$$

where $\rho > 0$ and n is a positive integer. For what value(s) of n is $\operatorname{div}\mathbf{f} = 0$ everywhere?

We have interpreted divergence in terms of net outflow rate from a point. Not all vector fields are associated with flow of material or energy. For example, there is no flow associated with a static magnetic field or an electrostatic field. However, the divergences of such fields can still have physical significance.

For example, the divergence of any magnetic field $\mathbf{B}$ is always zero. This law,

$$\operatorname{div}\mathbf{B} = 0, \tag{13}$$

is true for all possible magnetic fields and is an expression of the fact that there are no sources or sinks in a magnetic field.

This law is one of four equations called *Maxwell's equations*. It can be read as saying that the magnetic field $\mathbf{B}$ is a solenoidal vector field.

Exercise 10

Use equation (13) to decide which of the following vector fields could be magnetic fields:

$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + xy\mathbf{k}$,
$\mathbf{G}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$,
$\mathbf{H}(x, y, z) = x\mathbf{i} + y\mathbf{j} - 2z\mathbf{k}$.

2 Curl of a vector field

You have seen that the divergence of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ is a scalar field given by the sum of the three partial derivatives $\partial F_1/\partial x$, $\partial F_2/\partial y$ and $\partial F_3/\partial z$. This is one way of differentiating a vector field, resulting in a scalar. We now introduce the *curl* of a vector field $\mathbf{F}$, written $\mathbf{curl}\,\mathbf{F}$. This new field is a vector field constructed from the other six partial derivatives of F_1, F_2 and F_3 with respect to Cartesian coordinates x, y, z. You can think of these partial derivatives as 'sideways' derivatives. For example, $\partial F_1/\partial y$ describes how the x-component of $\mathbf{F}$ changes with small displacements in the y-direction. Subsections 2.1 and 2.2 introduce the curl of a vector field and show how to calculate it. Subsection 2.3 provides a physical interpretation of curl.

Note that 'curl' is emboldened in $\mathbf{curl}\,\mathbf{F}$ since it is a vector field; in handwritten work it is denoted using a straight or wavy underline.

2.1 Defining curl

In this subsection we consider the mathematical definition of $\mathbf{curl}\,\mathbf{F}$ using the vector differential operator $\boldsymbol{\nabla}$. You have seen that for any scalar field f and for any vector field $\mathbf{F}$, we have

$$\mathbf{grad}\, f = \boldsymbol{\nabla} f,$$
$$\operatorname{div}\mathbf{F} = \boldsymbol{\nabla}\cdot\mathbf{F}.$$

$\mathbf{grad}\, f$ is a vector field.

$\operatorname{div}\mathbf{F}$ is a scalar field.

We now define the vector field $\mathbf{curl}\,\mathbf{F}$ to be the cross product, $\boldsymbol{\nabla}\times\mathbf{F}$, of $\boldsymbol{\nabla} = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$ and $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$.

Curl

The **curl** of a vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ in **Cartesian coordinates** is

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \boldsymbol{\nabla} \times \mathbf{F} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \qquad (14)\end{aligned}$$

The determinant form of the cross product was given in Unit 4, Subsection 2.4.

The name 'curl' comes from the fact that the vector field **curl F** represents the 'local rotation' in the field **F**. Local rotation will be discussed in Subsection 2.3.

When the vector field $\mathbf{F}(x, y)$ is confined to the (x, y)-plane, F_3 is zero, and F_1 and F_2 do not depend on z. Then you can see from equation (14) that only the z-component of **curl F** can be non-zero, and we have the following simplified expression.

Curl of a two-dimensional vector field

The **curl of a two-dimensional vector field** $\mathbf{F}(x, y)$ is

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k}. \qquad (15)$$

This means that the curl of a two-dimensional vector field must either be the zero vector or be directed at right angles to the plane of the field.

Examples of both possibilities are given in Example 4.

2.2 Calculating curl

To calculate the curl of a given vector field, first identify the component fields, then evaluate the relevant partial derivatives, and substitute these partial derivatives into the appropriate expression for curl in equation (14) or equation (15).

Example 4

(a) Find the curl of each of the following vector fields.

(i) $\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j}$

(ii) $\mathbf{g}(x, y) = -y\mathbf{i} + x\mathbf{j}$

(iii) $\mathbf{F}(x, y, z) = (xy + z^2)\mathbf{i} + x^2\mathbf{j} + (xz - 2)\mathbf{k}$

(b) (i) Find $\mathbf{curl}\,\mathbf{F}(1,-1,3)$, where $\mathbf{F}$ is the vector field specified in part (a)(iii).

(ii) Find points where $\mathbf{curl}\,\mathbf{F}=\mathbf{0}$, where $\mathbf{F}$ is the vector field specified in part (a)(iii).

Solution

(a) (i) The components of $\mathbf{f}$ are $f_1=x$ and $f_2=y$, so

$$\frac{\partial f_1}{\partial y}=0, \quad \frac{\partial f_2}{\partial x}=0.$$

The vector field $f(x,y)$ is confined to the (x,y)-plane, so we can use the simplified version. Substituting in equation (15), we find $\mathbf{curl}\,\mathbf{f}=\mathbf{0}$.

We write the bold zero vector here since curl is a *vector* field.

(ii) The components of $\mathbf{g}$ are $g_1=-y$ and $g_2=x$, so

$$\frac{\partial g_1}{\partial y}=-1, \quad \frac{\partial g_2}{\partial x}=1.$$

Substituting in equation (15), we obtain

$$\mathbf{curl}\,\mathbf{g}=\left(\frac{\partial g_2}{\partial x}-\frac{\partial g_1}{\partial y}\right)\mathbf{k}=(1-(-1))\mathbf{k}=2\mathbf{k}.$$

(iii) We have $F_1=xy+z^2$, $F_2=x^2$, $F_3=xz-2$, so

$$\frac{\partial F_1}{\partial y}=x, \quad \frac{\partial F_1}{\partial z}=2z,$$
$$\frac{\partial F_2}{\partial x}=2x, \quad \frac{\partial F_2}{\partial z}=0,$$
$$\frac{\partial F_3}{\partial x}=z, \quad \frac{\partial F_3}{\partial y}=0.$$

Substituting in equation (14), we obtain

$$\mathbf{curl}\,\mathbf{F}=(0-0)\mathbf{i}+(2z-z)\mathbf{j}+(2x-x)\mathbf{k}=z\mathbf{j}+x\mathbf{k}.$$

(b) (i) At the point $(1,-1,3)$, we have $\mathbf{curl}\,\mathbf{F}=3\mathbf{j}+\mathbf{k}$.

(ii) When $x=0$ and $z=0$, and for any value of y, $\mathbf{curl}\,\mathbf{F}=\mathbf{0}$, that is, $\mathbf{curl}\,\mathbf{F}$ is zero everywhere on the y-axis.

Exercise 11

(a) Find $\mathbf{curl}\,\mathbf{F}$, where $\mathbf{F}(x,y,z)=x(y-z)\mathbf{i}+3x^2\mathbf{j}+yz\mathbf{k}$.

(b) Determine $\mathbf{curl}\,\mathbf{F}$ at:

(i) the origin

(ii) the point $(1,2,3)$

(iii) the point that is 5 units from the origin on the positive z-axis.

Exercise 12

Consider $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ and $\widehat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$. Find $\mathbf{curl}\,\mathbf{F}$ in each of the following cases.

(a) $\mathbf{F} = \mathbf{r}$

(b) $\mathbf{F} = f(r)\widehat{\mathbf{r}}$, where f is a function of r only

(c) $\mathbf{F} = (y^2 + 2z)\mathbf{i} + (xy + 6z)\mathbf{j} + (z^2 + 2xz + y)\mathbf{k}$

(d) $\mathbf{F} = \mathbf{grad}\, f$, where f is any differentiable scalar field

Exercise 13

Suppose that two vector fields, $\mathbf{A}$ and $\mathbf{B}$, are related by $\mathbf{B} = \mathbf{curl}\,\mathbf{A}$. Show that provided that $\mathbf{A}$ is sufficiently smooth, this implies that $\mathrm{div}\,\mathbf{B} = 0$ everywhere, that is, $\mathbf{B}$ is a solenoidal field.

If a vector field $\mathbf{F}$ has $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere, then it is said to be an **irrotational field**. Irrotational force fields are important in mechanics. If a force field acts on a particle such that the total mechanical energy of the particle, that is, potential energy + kinetic energy, is conserved, then the force field will be irrotational. More will be said about this in Section 4.

Exercise 14

Identify irrotational fields from those considered in Example 4 and Exercise 12.

Exercise 15

Show that any vector field of the form $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$, where f is a function of x only and g is a function of y only, is irrotational.

2.3 Curl and local rotation

Whereas the divergence of a vector field represents the outflow rate from a point, the curl of a vector field represents rotation at a point.

We are assuming that the flow has settled down so that there are no changes in time. It is steady flow.

Consider the two-dimensional velocity field on the surface of a river. Let the surface be in the (x, y)-plane with the coordinate axes fixed relative to the river bank. Then the water surface flows through fixed points (x, y) with velocity $\mathbf{v}(x, y)$. This surface velocity field can be revealed by watching a floating object such as a small leaf. Then the velocity $\mathbf{v}(x, y)$ is the velocity of the leaf as it passes through the fixed point (x, y).

If you watch a small leaf floating down a river, you may notice two types of motion. First, the leaf floats downstream following a vector field line of the velocity field $\mathbf{v}(x, y)$. Then the leaf may rotate in the plane of the surface. The rate of rotation may be quite fast for an object floating near the bank of a fast flowing river. However, near midstream, the rate of rotation is likely to be slow or zero, even when the surface velocity is fast.

The rotation of a floating body is easily explained. It happens whenever the velocity, and hence the drag, is greater on one side of the object than on the other. There is then a torque on the object, thus it rotates. Figure 5 shows a typical distribution of surface velocities $\mathbf{v}$ on a line across a river. You can see that the velocity magnitude increases quite rapidly as you move out from a river bank, but varies hardly at all near midstream. Hence the difference in drag on the near and far sides of a small floating object may be quite large near a bank and very small near midstream.

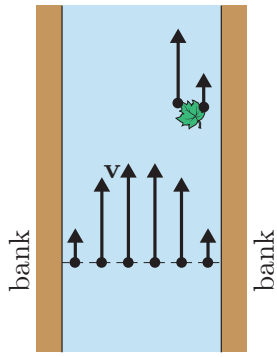


Figure 5 Velocity field on the surface of a river

A rotation about the z-axis is described by a vector $\boldsymbol{\omega} = \omega\mathbf{k}$, called the **angular velocity**. The magnitude $|\boldsymbol{\omega}| = \omega$ describes the angular speed, that is, the rate at which angle (such as the angle of orientation of the floating leaf on the river) changes in time during the rotation (measured in radians per second). The direction of the vector $\boldsymbol{\omega}$ is at right angles to the plane of rotation in the sense given by the right-hand grip rule. In the case of the rotating leaf on a river, the direction of the rotation is at right angles to the river surface, that is, parallel to the z-axis.

Rotation about an axis will be discussed in more detail in Unit 20.

See Unit 2 for the right-hand grip rule.

Perhaps you can now begin to see a relationship between the rotation of a small floating object, as described by a vector $\boldsymbol{\omega}$, and the curl of the surface velocity field **curl v**. The curl of any two-dimensional vector field on a surface is directed at right angles to the surface. Hence $\boldsymbol{\omega}$ and **curl v** are both vectors directed at right angles to the water surface. Furthermore, **curl v** is built up from the partial derivatives $\partial v_1/\partial y$ and $\partial v_2/\partial x$, which represent variation perpendicular to the directions of the corresponding component vectors $v_1\mathbf{i}$ and $v_2\mathbf{j}$, and so correspond to rotational motion. Furthermore, the angular speed $|\boldsymbol{\omega}|$ is largest where the *downstream* surface velocity varies most rapidly with distance *across* the river. In fact, the rotation of a small floating body can be modelled quite well by

$$\boldsymbol{\omega} = A\,\mathbf{curl}\,\mathbf{v},$$

where A is a positive constant that depends on the size and shape of the body, the nature of its surface and the properties of the fluid.

The above relationship applies also in three dimensions and is the basis of experimental techniques for exploring the curl of velocity fields in fluids by using a neutral density probe, that is, a small object that can float at a point beneath the surface of a fluid without sinking or rising. The magnitude and direction of the rotation of the floating probe provide an indication of the curl of the velocity at that point.

In the next example we consider the surface velocity field on a model river, calculate the curl of the velocity, and confirm that it describes the way in which a small floating object would rotate.

Example 5

The surface water velocity on a straight uniform river can often be modelled by a two-dimensional vector field of the form

$$\mathbf{v}(x, y) = Cx(d - x)\mathbf{j} \quad (0 \leq x \leq d),$$

where the y-axis is along one bank and points downstream, d is the width of the river, and C is a positive constant (see Figure 6).

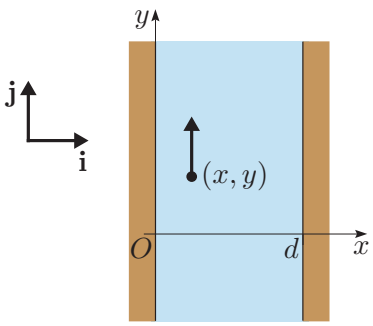


Figure 6 River with surface velocity $\mathbf{v}(x, y)$ and banks running along $x = 0$ and $x = d$

(a) What is the surface velocity at the river banks and at midstream? How does the magnitude of the surface velocity change as you move out from one bank towards midstream?

(b) Find the vector field **curl v**. State the magnitude and direction of **curl v** at the river banks and at midstream.

(c) Explain why the answers to part (b) are consistent with the idea that **curl v** describes the rate and sense of rotation of a small leaf floating on the river surface.

Solution

(a) The river banks are the lines $x = 0$ (the y-axis) and $x = d$. We find that $\mathbf{v}(0, y) = \mathbf{v}(d, y) = \mathbf{0}$, so the river velocity is the zero vector at the banks. At midstream, $x = \frac{1}{2}d$ and $\mathbf{v}(\frac{1}{2}d, y) = \frac{1}{4}Cd^2\mathbf{j}$. Hence the velocity at midstream is directed downstream (that is, up the page in Figure 6) and is of magnitude $\frac{1}{4}Cd^2$. The surface speed increases as you move out from either bank towards midstream.

(b) The components of $\mathbf{v}$ are $v_1 = 0$ and $v_2 = Cx(d - x)$. Hence, using equation (15), we have $\mathbf{curl}\,\mathbf{v} = (Cd - 2Cx)\mathbf{k} = C(d - 2x)\mathbf{k}$. The value of **curl v** at the bank $x = 0$ is $Cd\mathbf{k}$. The value of **curl v** at the bank $x = d$ is $-Cd\mathbf{k}$. The value of **curl v** at midstream ($x = \frac{1}{2}d$) is $\mathbf{0}$.

(c) We have found that at $x = 0$, **curl v** points in the positive z-direction (vertically upwards).

A small floating leaf near $x = 0$ would rotate anticlockwise as it floats downstream, since its outer edge is pulled faster than the inner edge by the flow.

Similarly, a leaf near the bank $x = d$ would rotate clockwise. This is in accordance with **curl v** being a vector pointing in the negative z-direction at $x = d$.

At midstream, the leaf would float without rotating, since both sides of the leaf are pulled equally by the flow. This corresponds to **curl v** $= \mathbf{0}$ at midstream.

This corresponds to an anticlockwise rotation by the right-hand grip rule of Unit 2; this will be explained in greater detail in Unit 20.

The rotation in the velocity field $\mathbf{v}$ described by **curl** $\mathbf{v}$ is the **local rotation** in the field, as measured by the rotation of a small floating object. Local rotation should be distinguished from **bulk rotation**, which is simply the movement of material around a circular path.

There is local rotation, but no bulk rotation, in Example 5, since the flow is parallel to the y-direction everywhere. There is bulk rotation of water in the velocity fields $\mathbf{u}$ and $\mathbf{v}$ that model the surface velocity of water in an emptying bathtub in Exercise 16 below, but you will find that there is local rotation in only one of them.

Exercise 16

The velocity field on the surface of an emptying bathtub can be modelled by two functions, the first describing the vigorously swirling vortex of radius a in a central region, and the second describing the more gently rotating fluid outside the vortex region. These functions are

$$\mathbf{u}(x,y) = \omega(-y\mathbf{i} + x\mathbf{j}) \quad (\sqrt{x^2+y^2} \leq a),$$
$$\mathbf{v}(x,y) = \frac{\omega a^2(-y\mathbf{i} + x\mathbf{j})}{x^2+y^2} \quad (\sqrt{x^2+y^2} \geq a),$$

Note that $\mathbf{u} = \mathbf{v}$ when $(x^2+y^2)^{1/2} = a$.

where ω is a positive constant and the water surface is assumed to lie in the (x,y)-plane.

Find **curl** $\mathbf{u}$ and **curl** $\mathbf{v}$.

Exercise 17

Express the two vector fields $\mathbf{u}$ and $\mathbf{v}$ of Exercise 16 in cylindrical coordinates. What shape are the vector field lines of $\mathbf{u}$ and $\mathbf{v}$?

Both velocity fields $\mathbf{u}$ and $\mathbf{v}$ in Exercise 16 have a bulk rotation of fluid around a central point, but field $\mathbf{v}$ has **curl** $\mathbf{v} = \mathbf{0}$, showing that there is no local rotation in the field $\mathbf{v}$. A small floating object in the region of the field $\mathbf{v}$ would float around a circular field line, but would not rotate, that is, it would maintain the same orientation in space.

If you try this experiment next time you have a bath, you will probably see some rotation. This is hardly surprising, since the field $\mathbf{v}$ is a simple model that assumes that the flow is circular, whereas in reality the flow *spirals into* the plughole.

In Exercise 17 you found that

$$\mathbf{u}(\rho,\phi,z) = \omega\rho\,\mathbf{e}_\phi \quad (\rho \leq a),$$
$$\mathbf{v}(\rho,\phi,z) = \frac{\omega a^2}{\rho}\,\mathbf{e}_\phi \quad (\rho \geq a).$$

Hence the fields $\mathbf{u}$ and $\mathbf{v}$ have simple forms when expressed in cylindrical coordinates. When we are given a vector field in cylindrical coordinates, it is usually easier to calculate the curl using the cylindrical form of curl rather than converting into Cartesian coordinates.

Exercise 18

The Earth rotates about its north–south axis at the rate of 2π radians per 24 hours. This rotation has angular velocity $\boldsymbol{\omega} = \omega\mathbf{k}$, where the unit vector $\mathbf{k}$ points from south to north, and $\omega = |\boldsymbol{\omega}|$ is the angular speed. The velocity of a point inside the Earth or on the Earth's surface is given by

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \quad (|\mathbf{r}| \leq R),$$

where $\mathbf{r}$ is the position vector of the point measured from an origin O at the Earth's centre, and R is the Earth's radius.

(a) Express the vector field $\mathbf{v}$ in Cartesian coordinates, taking the z-axis to be directed from south to north along the axis of rotation.

(b) Find $\mathbf{curl}\,\mathbf{v}$ and determine its magnitude.

(c) Comment on the relationship between the magnitude and direction of $\mathbf{curl}\,\mathbf{v}$, and the rate and sense of rotation of your home.

Curl in cylindrical and spherical coordinates

We will derive the formula for curl only in cylindrical coordinates. (The method for spherical coordinates is similar, but the calculations are more complicated.)

First recall, from Subsection 4.1 of Unit 15, that in cylindrical coordinates the gradient vector is given by

$$\boldsymbol{\nabla} = \mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}.$$

To compute $\mathbf{curl}\,\mathbf{F} = \boldsymbol{\nabla} \times \mathbf{F}$, we must therefore calculate

$$\left(\mathbf{e}_\rho \frac{\partial}{\partial \rho} + \mathbf{e}_\phi \frac{1}{\rho} \frac{\partial}{\partial \phi} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \times (F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z),$$

and since this is a complicated expression, it is sensible to proceed term by term.

The cross product of any vector with itself is $\mathbf{0}$, and

$\mathbf{e}_\rho \times \mathbf{e}_\phi = \mathbf{e}_z$,
$\mathbf{e}_\phi \times \mathbf{e}_z = \mathbf{e}_\rho$,
$\mathbf{e}_z \times \mathbf{e}_\rho = \mathbf{e}_\phi$.

Since $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ do not depend on ρ, the first term is

$$\begin{aligned}
\mathbf{e}_\rho \frac{\partial}{\partial \rho} \times (F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z) &= \mathbf{e}_\rho \times \frac{\partial}{\partial \rho}(F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z) \\
&= \mathbf{e}_\rho \times \left(\frac{\partial F_\rho}{\partial \rho}\,\mathbf{e}_\rho + \frac{\partial F_\phi}{\partial \rho}\,\mathbf{e}_\phi + \frac{\partial F_z}{\partial \rho}\,\mathbf{e}_z\right) \\
&= \frac{\partial F_\phi}{\partial \rho}\,\mathbf{e}_z - \frac{\partial F_z}{\partial \rho}\,\mathbf{e}_\phi.
\end{aligned}$$

Next, using the results $\partial \mathbf{e}_\rho/\partial\phi = \mathbf{e}_\phi$, $\partial \mathbf{e}_\phi/\partial\phi = -\mathbf{e}_\rho$ and $\partial \mathbf{e}_z/\partial\phi = \mathbf{0}$, we have

$$\begin{aligned}
&\mathbf{e}_\phi \frac{1}{\rho}\frac{\partial}{\partial\phi} \times (F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z) \\
&= \frac{1}{\rho}\,\mathbf{e}_\phi \times \frac{\partial}{\partial\phi}\,(F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z) \\
&= \frac{1}{\rho}\,\mathbf{e}_\phi \times \left(\frac{\partial F_\rho}{\partial\phi}\,\mathbf{e}_\rho + F_\rho \mathbf{e}_\phi + \frac{\partial F_\phi}{\partial\phi}\,\mathbf{e}_\phi - F_\phi \mathbf{e}_\rho + \frac{\partial F_z}{\partial\phi}\,\mathbf{e}_z\right) \\
&= -\frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi}\,\mathbf{e}_z + \frac{1}{\rho}F_\phi\,\mathbf{e}_z + \frac{1}{\rho}\frac{\partial F_z}{\partial\phi}\,\mathbf{e}_\rho.
\end{aligned}$$

Finally, since $\mathbf{e}_\rho$, $\mathbf{e}_\phi$ and $\mathbf{e}_z$ do not depend on z,

$$\begin{aligned}
\mathbf{e}_z \frac{\partial}{\partial z} \times (F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z) &= \mathbf{e}_z \times \frac{\partial}{\partial z}(F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z) \\
&= \mathbf{e}_z \times \left(\frac{\partial F_\rho}{\partial z}\,\mathbf{e}_\rho + \frac{\partial F_\phi}{\partial z}\,\mathbf{e}_\phi + \frac{\partial F_z}{\partial z}\,\mathbf{e}_z\right) \\
&= \frac{\partial F_\rho}{\partial z}\,\mathbf{e}_\phi - \frac{\partial F_\phi}{\partial z}\,\mathbf{e}_\rho.
\end{aligned}$$

Hence the formula, which we summarise below. A more complicated derivation provides a corresponding formula in spherical coordinates.

Curl in cylindrical and spherical coordinates

The **curl** of a vector field $\mathbf{F} = \mathbf{F}(\rho, \phi, z) = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi + F_z \mathbf{e}_z$ in **cylindrical coordinates** is

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} = &\left(\frac{1}{\rho}\frac{\partial F_z}{\partial\phi} - \frac{\partial F_\phi}{\partial z}\right)\mathbf{e}_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\mathbf{e}_\phi \\
&+ \left(\frac{\partial F_\phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial\phi} + \frac{1}{\rho}F_\phi\right)\mathbf{e}_z. \qquad (16)
\end{aligned}$$

The curl of a vector field $\mathbf{F}(r, \theta, \phi) = F_r \mathbf{e}_r + F_\theta \mathbf{e}_\theta + F_\phi \mathbf{e}_\phi$ in **spherical coordinates** is

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F} = &\left(\frac{1}{r}\frac{\partial F_\phi}{\partial\theta} - \frac{1}{r\sin\theta}\frac{\partial F_\theta}{\partial\phi} + \frac{\cot\theta}{r}F_\phi\right)\mathbf{e}_r \\
&+ \left(-\frac{\partial F_\phi}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}F_\phi\right)\mathbf{e}_\theta \\
&+ \left(\frac{\partial F_\theta}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial\theta} + \frac{1}{r}F_\theta\right)\mathbf{e}_\phi. \qquad (17)
\end{aligned}$$

As you can see, the full cylindrical expression for curl is quite complicated. However, for vector fields where there is no z-component and no variation in the z-direction, we have the following.

Curl of a two-dimensional vector field

The **curl** of a two-dimensional vector field $\mathbf{F} = \mathbf{F}(\rho, \phi) = F_\rho \mathbf{e}_\rho + F_\phi \mathbf{e}_\phi$ in **cylindrical coordinates** is

$$\boldsymbol{\nabla} \times \mathbf{F} = \left(\frac{\partial F_\phi}{\partial \rho} + \frac{F_\phi}{\rho} - \frac{1}{\rho} \frac{\partial F_\rho}{\partial \phi} \right) \mathbf{e}_z. \tag{18}$$

Example 6

Use equation (18) to calculate **curl u** and **curl v** where **u** and **v** are the velocity fields in cylindrical form that you found in Exercise 17.

Solution

For the field **u**, we have $u_\rho = 0$, $u_\phi = \omega\rho$. The only non-zero partial derivative is $\partial u_\phi / \partial \rho = \omega$. Hence

$$\mathbf{curl}\, \mathbf{u} = \left(\omega + \frac{\omega\rho}{\rho} \right) \mathbf{e}_z = 2\omega \mathbf{e}_z = 2\omega \mathbf{k}.$$

Similarly, we have $v_\rho = 0$, $v_\phi = \omega a^2/\rho$, so the only non-zero partial derivative for **v** is $\partial v_\phi / \partial \rho = -\omega a^2/\rho^2$. Hence

$$\mathbf{curl}\, \mathbf{v} = \left(-\frac{\omega a^2}{\rho^2} + \frac{\omega a^2}{\rho^2} \right) \mathbf{e}_z = \mathbf{0}.$$

These results agree with those of Exercise 16.

Exercise 19

Consider the vector field **F** expressed in cylindrical coordinates as

$$\mathbf{F}(\rho, \phi) = f(\rho)\, \mathbf{e}_\rho + \rho^n g(\phi)\, \mathbf{e}_\phi, \quad \rho > 0,$$

where $f(\rho)$ is a function of ρ only, $g(\phi)$ is a function of ϕ only that is in general non-zero, and n is an integer.

Show that **F** is irrotational if and only if $n = -1$.

Exercise 20

Consider the vector field $\mathbf{F} = f(r)\, \widehat{\mathbf{r}}$, where $r = |\mathbf{r}|$, $\widehat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ (the position vector), and $f(r)$ is a function of r only.

Use the expression for curl in spherical coordinates to show that **F** is irrotational.

Exercise 21

Consider the vector field $\mathbf{A}$ expressed in spherical coordinates as

$$\mathbf{A}(r,\theta,\phi) = \frac{\psi(r,\theta)}{r\sin\theta}\,\mathbf{e}_\phi, \quad r>0,\ 0<\theta<\pi,$$

where $\psi(r,\theta)$ does not depend on ϕ, and let a second vector field, $\mathbf{F}$, be given by $\mathbf{F} = \widehat{\mathbf{r}}/r^2 = \mathbf{e}_r/r^2$ (in spherical coordinates).

Show that $\mathbf{F} = \mathbf{curl}\,\mathbf{A}$ provided that the function $\psi(r,\theta)$ satisfies

$$\frac{\partial\psi}{\partial r} = 0, \quad \frac{\partial\psi}{\partial\theta} = \sin\theta.$$

Hence deduce that the family of functions that satisfy these requirements is given by $\psi(r,\theta) = c - \cos\theta$, for some arbitrary constant c.

3 The scalar line integral

Scalar line integrals occur whenever we sum scalar values along a line or along a curve in space. An important example of a scalar line integral is the *work done by a force*. Subsection 3.1 defines the work done by a force acting on a particle moving along the x-axis. This definition is then generalised to motion along a curve in space. In Subsection 3.2 we show that the work done by the force is a *scalar line integral* of the force vector along the curve, and we also show how scalar line integrals are evaluated. In Subsection 3.3 we show how the length of a curve can be expressed as a scalar line integral.

We often refer to scalar line integrals as just *line integrals*.

3.1 Work done by forces

We begin with a one-dimensional problem in which a particle is acted on by a single force $\mathbf{F}$. Suppose that a constant force $F\mathbf{i}$ accelerates a particle of mass m in the direction of the positive x-axis. If $a_0\mathbf{i}$ is the constant acceleration of the particle, $x_0\mathbf{i}$ is its initial position, and v_0 is its initial speed, then the position $x\mathbf{i}$ and velocity $v\mathbf{i}$ of the particle are related by the constant acceleration equation

You met problems of this sort in Unit 3. If F is constant, then so is the acceleration $a_0\mathbf{i}$, and $F = ma_0$.

$$v^2 = v_0^2 + 2a_0(x - x_0).$$

See Subsection 1.2 of Unit 3.

Multiplying by $\frac{1}{2}m$ and rearranging gives the energy conservation equation

$$\tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_0^2 = ma_0(x - x_0) = F(x - x_0). \tag{19}$$

In Unit 9 an equation of this type was interpreted as stating that the gain (or loss) of kinetic energy equals the loss (or gain) of potential energy. Here we are interested in two other ways of interpreting equation (19).

First, since both the force and the displacement of the particle are directed along the x-axis, the right-hand side is just the dot product of $\mathbf{F} = F\mathbf{i}$ and $(x - x_0)\mathbf{i}$, so

$$ma_0(x - x_0) = F(x - x_0) = \mathbf{F} \cdot (x - x_0)\mathbf{i}.$$

Second, since F is a constant and the particle is moving in a straight line from x_0 to a general position x, we can write

$$F(x - x_0) = \int_{x_0}^{x} F\,dx, \tag{20}$$

so the right-hand side of equation (19) is just the integral of F, along the x-axis, from x_0 to x. We call this integral the *work done* by $\mathbf{F}$ in moving the particle along the straight line from x_0 to x.

Here $h = x - x_0$, $a_0 = g$ and $v_0 = 0$ in equation (19).

We ignore all other forces such as air resistance acting on the stone.

You can see from this that the work done by the force is equal to the *gain* in kinetic energy of the particle. For a stone falling a distance h from rest, the constant force of gravity, $F_g\mathbf{i} = mg\mathbf{i}$, does work $F_g h = mgh$, which is equal to the *loss* of gravitational potential energy. The work done by the force is equal to the amount of gravitational potential energy that becomes kinetic energy.

We now generalise the work integral (20) to motion that is no longer confined to the x-axis but is either two-dimensional, along a path in a plane, or in three-dimensional space. In general, the force $\mathbf{F}$ will act at points $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in space, and we use the notation $\mathbf{F}(\mathbf{r})$ instead of $\mathbf{F}(x, y, z)$. We calculate the work done by any force field $\mathbf{F}(\mathbf{r})$, which depends at most on the position vector $\mathbf{r}$. Thus we exclude, for example, forces that depend on velocity, such as friction, or forces that depend explicitly on time as well as position.

We start with Newton's second law, $m\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r})$, for a particle of mass m. We take the dot product of both sides of the equation with the velocity $\dot{\mathbf{r}} = d\mathbf{r}/dt$ to give $m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \mathbf{F}(\mathbf{r}) \cdot \dot{\mathbf{r}}$. Since $d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/dt = 2\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}}$, this is equivalent to $\frac{1}{2}m\,d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})/dt = \mathbf{F}(\mathbf{r}) \cdot \dot{\mathbf{r}}$. We then integrate both sides of the equation with respect to time over the interval from t_0 to t_1. Thus, with $\mathbf{v}_0 = \dot{\mathbf{r}}(t_0)$ and $\mathbf{v}_1 = \dot{\mathbf{r}}(t_1)$, we have

$$\begin{aligned}\int_{t_0}^{t_1} \left(\tfrac{1}{2}m\,\frac{d(\dot{\mathbf{r}} \cdot \dot{\mathbf{r}})}{dt}\right) dt &= \tfrac{1}{2}m\dot{\mathbf{r}}(t_1) \cdot \dot{\mathbf{r}}(t_1) - \tfrac{1}{2}m\dot{\mathbf{r}}(t_0) \cdot \dot{\mathbf{r}}(t_0) \\ &= \tfrac{1}{2}m\mathbf{v}_1 \cdot \mathbf{v}_1 - \tfrac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0.\end{aligned}$$

Since it is based on Newton's second law, equation (21) is valid only when $\mathbf{F}$ is the resultant force on the particle. However, when there is more than one force acting simultaneously, we can still define the *work done by each force separately* by an integral of the form of equation (22).

Hence we obtain a generalised form of equation (19),

$$\tfrac{1}{2}m\mathbf{v}_1 \cdot \mathbf{v}_1 - \tfrac{1}{2}m\mathbf{v}_0 \cdot \mathbf{v}_0 = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt, \tag{21}$$

where the left-hand side is the change of kinetic energy, and the definite integral

$$W = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt \tag{22}$$

is the **work done** by the force $\mathbf{F}$ on the particle when it moves along a path described by the position vector $\mathbf{r}(t)$, in the interval from t_0 to t_1.

Consider again the case of a particle moving in the positive direction of the x-axis, so that $\mathbf{r} = x\mathbf{i}$ and $\dot{\mathbf{r}} = \dot{x}\mathbf{i}$, but now we allow the magnitude of $\mathbf{F}$ to vary, so $\mathbf{F} = F(x)\,\mathbf{i}$. Suppose that the particle is at $x = a$ at time t_0, and at $x = b$ at time t_1, as shown in Figure 7. Since the force $\mathbf{F}$ depends at most on position x, and not on other variables such as velocity, or explicitly on time, the work done by the force, from equation (22), is

The case of a varying force accelerating a particle along the x-axis was considered in Section 3 of Unit 9.

$$W = \int_{t_0}^{t_1} \mathbf{F}(\mathbf{r}) \cdot \frac{d\mathbf{r}}{dt}\,dt = \int_{t_0}^{t_1} F(x)\,\mathbf{i} \cdot \frac{dx}{dt}\mathbf{i}\,dt$$
$$= \int_{t_0}^{t_1} F(x)\,\frac{dx}{dt}\,dt$$
$$= \int_a^b F(x)\,dx. \qquad (23)$$

Figure 7 Path of a particle moving in one dimension

Example 7

A particle moves along the x-axis from $x = 2$ to $x = 5$ under the action of a force $F(x)\,\mathbf{i}$ given by

$$F(x) = -100(x - 1).$$

Determine the work done by the force, and the change in the kinetic energy of the particle. (Assume that the initial kinetic energy of the particle is sufficiently high for the motion to be possible.)

We use SI units.

Solution

To calculate the work done, we can use equation (23), obtaining

$$W = \int_a^b F(x)\,dx = -100\int_2^5 (x - 1)\,dx = -100\left[\tfrac{1}{2}x^2 - x\right]_2^5 = -750.$$

Hence the work done by the force is $-750\,\mathrm{J}$ as the particle moves from $x = 2$ to $x = 5$.

The SI units of work (work done) are the same as those of energy (joules).

Assuming that $F(x)\,\mathbf{i}$ is the resultant force acting on the particle, the change in the particle's kinetic energy is $-750\,\mathrm{J}$. Note that the kinetic energy is reduced because the force always acts in the opposite direction to the direction of motion.

Exercise 22

The force on a particle of mass m moving on an interval of the x-axis is

$$F(x)\,\mathbf{i} = -\frac{A}{x^2}\,\mathbf{i} \quad (x \neq 0),$$

where A is a positive constant.

Determine the work done by the force when the particle moves from $x = 3$ to $x = 1$, and hence find an expression for the speed v of the particle when it is at $x = 1$, given that its speed at $x = 3$ was u.

Before we give the details of a general method for evaluating integrals like that in equation (22), for the work done by a force, let us first consider two special cases when the integrand, and hence the work done, is zero.

- $\mathbf{F} = \mathbf{0}$, which makes sense since if the force is zero, it can do no work.
- $\mathbf{F}$ and $d\mathbf{r}/dt$ are perpendicular, so $\mathbf{F} \cdot d\mathbf{r}/dt = 0$. Since the velocity vector $d\mathbf{r}/dt$ points along the tangent to the curve at each point, only the component of the force in this direction contributes to the work done. Hence no work is done if the only force is perpendicular to the direction of motion.

See Unit 20.

An example of the second case is provided by a particle moving at constant speed along a circular path. In this case the force is directed along the radius, towards the centre; it therefore has zero component along the tangent to the circle at any point, and hence does no work. Other examples of situations where forces always act at right angles to the path of a moving particle, and hence do no work, are the tension force in the string of a simple pendulum and the normal reaction on an object moving on an inclined plane.

The parameter t (for time) appears in equation (22) because it is often a convenient way of parametrising the path of the particle. We could equally well describe the path in terms of another parameter, such as the distance s from the point where $\mathbf{r} = \mathbf{r}(t_0)$. For this reason we normally write equation (22) symbolically in parameter-free form as

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \tag{24}$$

where C is the path of the particle. However, when we need to evaluate equation (24), we express the path in terms of a suitable **parameter**, usually t (which may or may not represent time), so that it once again takes the form of equation (22).

To evaluate equation (22) with the dot product in the integrand expressed in Cartesian form, we need to know the functions $\mathbf{F}(\mathbf{r})$ and $d\mathbf{r}/dt$. Now for $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, we have

You considered t as a parameter for the position vector $\mathbf{r}$ in Unit 3.

$$\frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k},$$

and we can express the components of $\mathbf{F}(\mathbf{r})$ at any point on C as functions of t. For example,

$$F_1(\mathbf{r}) = \mathbf{F}(\mathbf{r}) \cdot \mathbf{i} = F_1(x, y, z) = F_1(x(t), y(t), z(t)) = F_1(t),$$

so

$$\mathbf{F}(t) = F_1(t)\,\mathbf{i} + F_2(t)\,\mathbf{j} + F_3(t)\,\mathbf{k} \quad \text{on } C.$$

Notice that in employing a parameter, we also abuse notation by reducing a function of three coordinates $\mathbf{F}(\mathbf{r})$ to a function of one parameter $\mathbf{F}(t)$.

Thus the integrand in equation (22) is

$$\mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} = F_1(t)\,\frac{dx}{dt} + F_2(t)\,\frac{dy}{dt} + F_3(t)\,\frac{dz}{dt}.$$

Work done by a force

The work done by a force $\mathbf{F}(\mathbf{r})$, given in Cartesian coordinates, in moving a particle along a path C given by $\mathbf{r} = \mathbf{r}(t)$ from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$ is

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$$
$$= \int_{t_0}^{t_1} \left(F_1(t) \frac{dx}{dt} + F_2(t) \frac{dy}{dt} + F_3(t) \frac{dz}{dt} \right) dt. \qquad (25)$$

Thus we have expressed the work done by a force as an ordinary definite integral that we can try to evaluate by standard integration techniques.

Example 8

Find the work done by the force

$$\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$$

when it moves a particle from the point $(0, 0, 0)$ to the point $(3, 1, 1)$ along the curve C given by $\mathbf{r} = 3t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$, from $t_0 = 0$ to $t_1 = 1$.

Solution

The components of $\mathbf{r}$ in terms of t, that is, the parametric equations for C, are

$$x = 3t, \quad y = t^2, \quad z = t^3, \quad \text{where } t \text{ goes from } t_0 = 0 \text{ to } t_1 = 1,$$

and differentiating these equations yields

$$\frac{dx}{dt} = 3, \quad \frac{dy}{dt} = 2t, \quad \frac{dz}{dt} = 3t^2.$$

The components of $\mathbf{F}$ in terms of t are

$$F_1 = yz = t^5, \quad F_2 = zx = 3t^4, \quad F_3 = xy = 3t^3.$$

Substituting into equation (25) gives

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$
$$= \int_0^1 \left((t^5)(3) + (3t^4)(2t) + (3t^3)(3t^2) \right) dt$$
$$= \int_0^1 18t^5 \, dt$$
$$= 18 \left[\tfrac{1}{6} t^6 \right]_0^1$$
$$= 3.$$

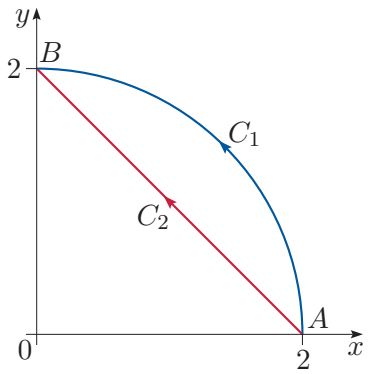


Figure 8 Two different paths from A to B

Exercise 23

Find the work done by the force

$$\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$$

as it moves a particle from the point $(2, 0)$ to the point $(0, 2)$ along the quarter-circle C_1 shown in Figure 8. The quarter-circle has centre at the origin and radius 2, and can be parametrised by the equations

$$x = 2\cos t, \quad y = 2\sin t, \quad z = 0, \quad \text{where } t \text{ goes from } t_0 = 0 \text{ to } t_1 = \tfrac{\pi}{2}.$$

Sometimes a curve in the (x, y)-plane is specified by an equation $y = f(x)$. We can then effectively use x as the parameter t in equation (25).

Example 9

Find the work done by the vector field $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ acting along the straight line C_2, specified by the equation $y = 2 - x$, from the point $(2, 0)$ to the point $(0, 2)$ (see Figure 8).

The force $\mathbf{F}$ is the same as that in Exercise 23.

Solution

We put $x = t$, and the parametric equations for C_2 are

$$x = t, \quad y = 2 - t, \quad z = 0,$$

where t runs from $t_0 = 2$ to $t_1 = 0$ (see Figure 8). Then we have

$$F_1 = 2x + y = 2 + t, \quad F_2 = -x = -t,$$
$$\frac{dx}{dt} = 1, \quad \frac{dy}{dt} = -1.$$

Substituting into equation (25) yields

$$\begin{aligned} W = \int_{C_2} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_2^0 \big((2 + t) + (-t)(-1)\big)\, dt \\ &= \int_2^0 (2t + 2)\, dt \\ &= \big[t^2 + 2t\big]_2^0 \\ &= -8. \end{aligned}$$

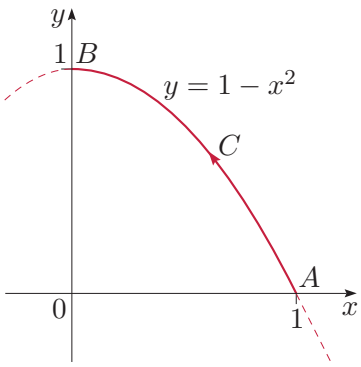


Figure 9

Exercise 24

Find the work done by the force $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$ acting along the parabolic segment C specified by the equation $y = 1 - x^2$, from the point $(1, 0)$ to the point $(0, 1)$ (see Figure 9).

Of course, there are many paths joining any two given points.

Exercise 25

Find two different paths between the points $(0, 0, 0)$ and $(1, 1, 1)$, giving parametrisations for each.

A parametrisation of a particular curve is not unique, nor does the work done along a particular path depend on the parametrisation.

Exercise 26

The straight line C_2 in Example 9 can be parametrised by the equations

$$x = 2(1 - t), \quad y = 2t, \quad z = 0, \quad \text{for } t \text{ from 0 to 1.}$$

Find the work done by the force $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ using this parametrisation, and compare your answer with that found in Example 9.

3.2 Scalar line integrals

You have seen that the work integral in equation (22) may be written as

$$W = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}, \tag{26}$$

This is equation (24).

where C specifies the path (or directed curve) along which the particle moves, and $\mathbf{r}$ is the position vector of points on C. You can think of the dot product between the force $\mathbf{F}(\mathbf{r})$ and the symbol $d\mathbf{r}$ in equation (26) as representing the fact that we are integrating only the component of the force parallel to the tangent to the curve in the direction of motion. This is illustrated in Figure 10, which shows a very short segment PQ of the curve C. The displacement $\delta\mathbf{r}_i = \mathbf{r}_Q - \mathbf{r}_P$ is nearly parallel to the tangents to the path along this segment, and the work done by the force along PQ is approximately the dot product $\mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{r}_i$, where $\mathbf{F}(\mathbf{r}_i)$ is the force vector at some point $\mathbf{r}_i$ on PQ. If we divide up C into N such short segments, then adding the approximations for the work done along all N segments, we obtain an estimate for the work done by $\mathbf{F}$ as it acts along the whole of C, that is,

$$W \simeq \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{r}_i.$$

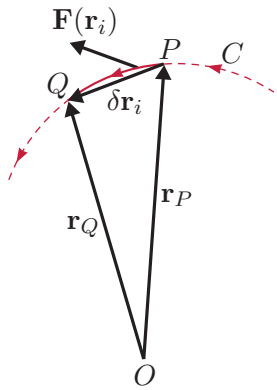


Figure 10 Small segment of a path of a particle subject to a force $\mathbf{F}(\mathbf{r})$

The approximation becomes exact in the limit as $N \to \infty$ and each $|\delta\mathbf{r}_i| \to 0$, to give

$$W = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbf{F}(\mathbf{r}_i) \cdot \delta\mathbf{r}_i = \int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}.$$

The work integral, equation (22) or equation (26), is an example of a *scalar line integral* of a vector field. We can form the scalar line integral of any vector field $\mathbf{F}(\mathbf{r})$ along a path C.

Scalar line integral

The **scalar line integral** of a vector field $\mathbf{F}(\mathbf{r})$ along a path C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$, is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt}\, dt. \tag{27}$$

Note that t may be any parameter, not necessarily time.

The physical interpretation of the scalar line integral will depend on the nature of the particular vector field $\mathbf{F}(\mathbf{r})$ being considered. We will consider only the case of force fields, where the scalar line integral represents the work done by the force.

We indicate the direction by an arrow in diagrams, e.g. Figures 8 and 9.

The path C in a scalar line integral is a directed curve (with starting point A and endpoint B). It is often convenient to denote the scalar line integral by writing it as

$$\int_{AB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r},$$

where the order AB indicates the direction along the path.

We could choose to traverse the *same* path but in the opposite sense, starting at B and ending at A. This would reverse the direction of each displacement $\delta\mathbf{r}_i$ in Figure 10 and therefore change the sign of the scalar line integral. Thus we have

$$\int_{BA} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = -\int_{AB} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}. \tag{28}$$

Line integrals can be evaluated along segments of **open curves** where the starting point A and the endpoint B are distinct points, as in the examples above, or around **closed curves**. The straight line and quarter-circle of Figure 8 are examples of open curves. Suppose now that we reverse the direction of the straight-line segment in Figure 8. Then the reversed line segment together with the quarter-circle make the closed curve shown in Figure 11. We can traverse this closed curve in an anticlockwise sense, $APBQA$, by starting at point A, moving along the quarter-circle via P to B, and returning along the straight line BA via Q. The line integral of a vector field $\mathbf{F}$ for one complete anticlockwise traversal of this loop is the sum of the two line integrals. We can write this as

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} + \int_{BQA} \mathbf{F} \cdot d\mathbf{r},$$

where the circle on the integral sign indicates that the path C is closed. The sense of each of the two line integrals on the right-hand side of the above equation, that is, the direction along the path, is indicated by the order of the letters APB and BQA.

Closed curves are also referred to as *loops*.

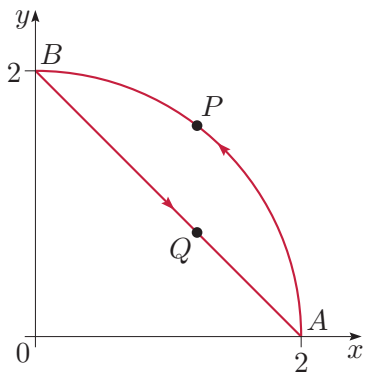


Figure 11 Path consisting of a closed curve $APBQA$ obtained from the paths in Figure 8 by combining path C_1 with the reverse of path C_2

Example 10

Evaluate

$$\oint_C \mathbf{F} \cdot d\mathbf{r},$$

where $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ and C is the closed curve $APBQA$ shown in Figure 11.

Solution

We sum the line integrals along the two segments APB and BQA of the loop. The line integral of $\mathbf{F}$ along the quarter-circle C_1 $(= APB)$ was evaluated in the solution to Exercise 23, where you obtained

$$\int_{APB} \mathbf{F} \cdot d\mathbf{r} = -4 - 2\pi.$$

The line integral along the straight line C_2 $(= AQB)$ was evaluated in Example 9, where we obtained

$$\int_{AQB} \mathbf{F} \cdot d\mathbf{r} = -8.$$

But we want the line integral along BQA, that is, along the same straight-line segment but in the opposite direction. Using equation (28), we just change the sign, to obtain

$$\int_{BQA} \mathbf{F} \cdot d\mathbf{r} = 8.$$

Thus

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} + \int_{BQA} \mathbf{F} \cdot d\mathbf{r} = -4 - 2\pi + 8 = 4 - 2\pi.$$

Exercise 27

Evaluate the scalar line integral of the vector field $\mathbf{G} = x^2\mathbf{i} + y\mathbf{j}$ along each of the paths C_1 and C_2 specified in Figure 8. (Use the parametrisations of Exercise 23 and Example 9, respectively.) Evaluate also the scalar line integral of $\mathbf{G}$ around the closed curve $APBQA$ in Figure 11.

Exercise 28

Evaluate the scalar line integral of $\mathbf{F} = 2x\mathbf{i} + (xz - 2)\mathbf{j} + xy\mathbf{k}$ along the path C from the point $(0, 0, 0)$ to the point $(1, 1, 1)$ defined by the parametrisation

$$x = t, \quad y = t^2, \quad z = t^3 \quad (0 \leq t \leq 1).$$

Example 11

A force $\mathbf{F}$ acts along a path C of length L. The tangential component F_t of $\mathbf{F}$ is constant along C. Show that the work done by $\mathbf{F}$ is $F_\mathrm{t}L$.

Solution

The component of $\mathbf{F}$ normal to C does no work since it is perpendicular to the tangent to C. So if the parameter l measures the length along C from $l_0 = 0$ to $l_1 = L$, and $F_\mathrm{t} = \mathbf{F} \cdot d\mathbf{r}/dl$, then the work done, W, is given by

$$W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{l_0}^{l_1} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dl} \right) dl = \int_{l_0}^{l_1} F_\mathrm{t}\, dl = \int_0^L F_\mathrm{t}\, dl = F_\mathrm{t}L.$$

Exercise 29

Evaluate the scalar line integral of each of the two vector fields $\mathbf{u}$ and $\mathbf{v}$ given below, around the closed circular path C of radius a centred on the origin and defined by the parametric equations

$$x = a\cos t, \quad y = a\sin t, \quad \text{where } t \text{ goes from 0 to } 2\pi.$$

(a) $\mathbf{u}(x, y) = (x\mathbf{i} + y\mathbf{j})/(x^2 + y^2)$

(b) $\mathbf{v}(x, y) = \omega(-y\mathbf{i} + x\mathbf{j})$, where ω is a positive constant

You may have spotted that the line integrals in Exercise 29 can be evaluated without explicitly carrying out an integration, since in each case the tangential component of the vector field is constant everywhere on the circle. This is obvious in part (a), since $x\mathbf{i} + y\mathbf{j} = \mathbf{r}$ is the position vector, which points radially outwards everywhere, and is therefore always perpendicular to the tangent to any circle centred on the origin. Hence the tangential component of $\mathbf{u}$ is zero everywhere on the circle, and the line integral is zero. It is not so obvious in part (b), but you can see that $\mathbf{r} \cdot \mathbf{v} = 0$, so the field $\mathbf{v}$ is at right angles to the position vector and is therefore directed tangentially to the path. Also, the field $\mathbf{v}$ has a constant magnitude, $\omega((-y)^2 + x^2)^{1/2} = \omega a$, on the circle. Thus the line integral of $\mathbf{v}$ for one complete traversal of the circle is $\omega a \times 2\pi a = 2\omega\pi a^2$. When evaluating line integrals, you can sometimes spot that the tangential component of the vector field is constant and, as this example shows, obtain the value directly using the result of Example 11.

The field lines of $\mathbf{v}$ are circles centred on the origin.

You might also note that on the circle of radius a we have $\mathbf{v} = \omega a\mathbf{e}_\theta$, where $\mathbf{e}_\theta$ is the unit vector in the direction of increasing θ in polar coordinates.

Exercise 30

Evaluate the scalar line integral of the vector field $\mathbf{v}$ defined in Exercise 29 along the x-axis from $x = 1$ to $x = -1$.

Exercise 31

Determine each of the following line integrals, which you can do without explicitly carrying out an integration.

(a) The line integral of $\mathbf{F} = z^2\mathbf{j}$ along the x-axis from $x = 1$ to $x = 2$

(b) The line integral of $\mathbf{F} = 5\mathbf{k}$ along the z-axis from $z = 0$ to $z = 6$

(c) The line integral of $\mathbf{F} = r^2\mathbf{e}_\theta$ on a semicircle in the (x, y)-plane centred on the origin and of radius 3

$\mathbf{F}$ is given here in polar coordinates.

3.3 The length of a curve

Consider the scalar line integral of the vector field $\mathbf{F}$ along the curve C given by $\mathbf{r} = \mathbf{r}(t)$, from $\mathbf{r}(t_0)$ to $\mathbf{r}(t_1)$, defined by

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt}\, dt,$$

where t represents time. The expression $\mathbf{r}(t)$ represents the point on the curve corresponding to time t, and $d\mathbf{r}/dt$ represents the velocity of the point as it moves along the curve. Now consider what happens when we choose the vector function

$$\mathbf{F}(t) = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} = \widehat{\dot{\mathbf{r}}(t)},$$

which represents a unit vector in the direction of the velocity vector $\dot{\mathbf{r}}(t)$. In this case

$$\mathbf{F}(t) \cdot \frac{d\mathbf{r}}{dt} = \frac{\dot{\mathbf{r}}(t)}{|\dot{\mathbf{r}}(t)|} \cdot \dot{\mathbf{r}}(t) = |\dot{\mathbf{r}}(t)|,$$

and the scalar line integral becomes

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_{t_0}^{t_1} |\dot{\mathbf{r}}(t)|\, dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \qquad (29)$$

This gives the length of the curve C between the two points $\mathbf{r}(t_0)$ and $\mathbf{r}(t_1)$. Indeed, one can see this intuitively by considering what happens to the point that is currently at $\mathbf{r}(t)$ during a small interval of time δt. During this time, the point will move along the curve by an approximate distance $|\dot{\mathbf{r}}(t)|\,\delta t$, so the length of the curve is approximated by the sum

$$\sum_i |\dot{\mathbf{r}}_i(t)|\,\delta t_i.$$

Allowing the intervals δt_i to shrink to zero in the usual way, we can replace the sum by an integral that is precisely equation (29). Let us check this in an example.

Example 12

Use the scalar line integral equation (29) and a suitable parametrisation to find the length of a semicircle of radius 1.

Solution

Take the centre of the circle to be at the origin, and consider the semicircle as lying in the upper half-plane ($y \geq 0$). Then a suitable parametrisation is given by $x = \cos t$, $y = \sin t$ $(0 \leq t \leq \pi)$. Thus $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$, and $\dot{\mathbf{r}}(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$.

Therefore

$$|\dot{\mathbf{r}}(t)| = 1,$$

and the integral in equation (29) is

$$\int_0^{\pi} 1\,dt = \pi,$$

which we know to be the correct answer.

Exercise 32

Find the length of the curve given by

$$x = t, \quad y = -\ln(\cos t) \quad (0 \leq t \leq \pi/4).$$

4 Linking line integrals, curl and gradient

In this section a connection is established between the properties of scalar line integrals of a vector field and some of the properties of scalar and vector fields that you met in Unit 15 and earlier in this unit, in particular the curl of the vector field. Subsection 4.2 shows how the properties of scalar line integrals are used for classifying vector fields as *conservative* or *non-conservative*.

4.1 Line integrals and curl

In this subsection we make important links between line integrals, the curl of a vector field, potential functions and the gradient of a scalar field. The curl of a vector field was defined in Section 2 in terms of partial derivatives. In fact, it is also possible to define the curl of a vector field in terms of line integrals. We do this below, and in the next subsection you will see that it leads to the *curl test* for conservative fields.

Figure 12 shows a closed curve C in the (x, y)-plane enclosing a point P. Let the area enclosed by the curve be A. Consider the line integral of an arbitrary vector field $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$ for one complete anticlockwise traversal of the curve, then divide the value of the line integral by the area inside the curve to obtain the quotient

$$Q = \frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r}.$$

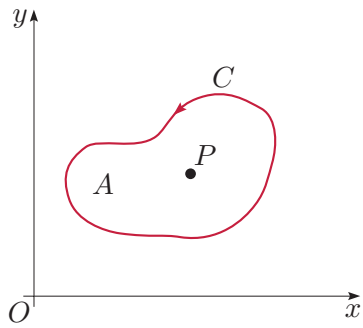


Figure 12 Closed curve with area A enclosing point P

In general, the value of Q might be expected to depend on the size and shape of the loop enclosing P. But suppose that we consider a sequence of loops of smaller and smaller area A, all enclosing P. We then find that the limit of Q for $A \to 0$ is independent of the shapes of the loops and of how the limit is approached, and depends only on the location of the point P, and of course the vector field $\mathbf{F}$. Furthermore – and here is a surprise – the limit turns out to be something familiar:

$$\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r}\right) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y},$$

where the partial derivatives are evaluated at P. (Note that F_3 does not appear here since $\mathbf{k}$ is perpendicular to the plane containing the loop C.) You will recognise the right-hand side as the z-component of $\mathbf{curl}\,\mathbf{F}$ at P. In other words, we have

$$\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{F} \cdot d\mathbf{r}\right) = \mathbf{k} \cdot \mathbf{curl}\,\mathbf{F}. \tag{30}$$

By considering loops in the (y, z)-plane and in the (z, x)-plane, we can obtain similar relationships for the x- and y-components of $\mathbf{curl}\,\mathbf{F}$.

These results are special cases of Stokes' theorem relating the curl of a vector field to the line integral of the field around a closed curve.

We will not prove these results for the general case, but ask you to confirm that equation (30) is true in a particular case.

Exercise 33

Consider the vector field $\mathbf{v} = \omega(-y\mathbf{i} + x\mathbf{j})$, where ω is a positive constant.

(a) Evaluate

$$\lim_{A\to 0}\left(\frac{1}{A}\oint_C \mathbf{v} \cdot d\mathbf{r}\right)$$

for anticlockwise traversal of the circle C of radius a centred on the origin.

The line integral is evaluated in the solution to Exercise 29(b).

(b) Evaluate $\mathbf{curl}\,\mathbf{v}$ and $\mathbf{k} \cdot \mathbf{curl}\,\mathbf{v}$.

From equation (30), and its counterparts for the x- and y-components, we can see that if a vector field $\mathbf{F}$ has the property that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C in the domain of $\mathbf{F}$, then $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere, that is, $\mathbf{F}$ is irrotational. However, the converse is not true *unless* the domain of $\mathbf{F}$ is simply-connected, a property that is defined as follows. A **simply-connected region** is a region where *any* closed curve contained in it can be continuously shrunk to a point without leaving the region.

Roughly speaking, a simply-connected region in two dimensions is one that contains no holes, and in three dimensions contains no holes that pass all the way though the region. For example, the (x, y)-plane is simply-connected, but the (x, y)-plane *excluding* the origin is *not*, for in this case it is not possible to continuously shrink a loop that encloses the origin to a point. Three-dimensional space $\mathbb{R}^3$ is simply-connected, as is $\mathbb{R}^3$ excluding the origin, but $\mathbb{R}^3$ excluding the z-axis is *not* simply-connected as loops that enclose the z-axis cannot be shrunk to a point.

Now, it can be shown that if $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ at all points in a vector field on a *simply-connected domain*, then $\oint_C \mathbf{F} \cdot d\mathbf{r}$ must be zero for any closed curve C within the domain, and vice versa. You will see later how to make use of this idea, but first we consider how to link line integrals with energy conservation, and the idea of a line integral being independent of the path along which it is evaluated. Suppose that a particle can move along paths in the two- or three-dimensional domain of a force field $\mathbf{F}(\mathbf{r})$. In general, a particle may move from one fixed point A to another fixed point B via an infinite number of different paths, so we might ask: does the work done by the force depend on the particular path taken from A to B? You will see that if the work done by the force is independent of the particular path, then we can define a potential energy function and mechanical energy is conserved.

In Example 9, we found that the line integral of the vector field $\mathbf{F} = (2x + y)\mathbf{i} - x\mathbf{j}$ on the straight line C_2, from the point $(2, 0)$ to the point $(0, 2)$, has the value -8; but in Exercise 23 you found that the line integral of the *same* vector field along the quarter-circle C_1 between the *same* two points $(2, 0)$ and $(0, 2)$ is $-4 - 2\pi$. In other words, the value of the line integral of the vector field between the two points depends on the actual path taken between the two points. The line integral of $\mathbf{F}$ is **path-dependent**. This is not surprising, since different paths between two fixed points will sample different vectors.

In Exercise 27, on the other hand, you found that the line integrals of the vector field $\mathbf{G} = x^2\mathbf{i} + y\mathbf{j}$ between the same two points have the same values for each of the two paths C_1 and C_2. In fact, the line integral of $\mathbf{G}$ between *any* two fixed points has the same value for *any* path between those two points. The line integral of $\mathbf{G}$ is **path-independent** and depends only on the starting point and endpoint of the path.

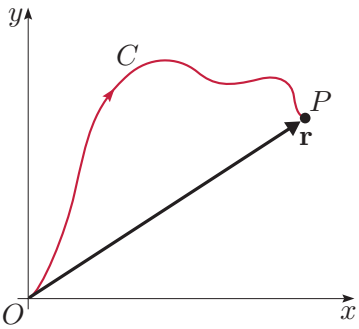


Figure 13 Path C joining O to P with position vector $\mathbf{r}$

We now turn to an idea that you met in Unit 15, that of the gradient vector. Suppose that the line integral of a vector field $\mathbf{F}(\mathbf{r})$ is path-independent. Given an origin O, we can define a scalar function $U(\mathbf{r})$ at every point P with position vector $\mathbf{r}$ in the domain of $\mathbf{F}$, by putting

$$U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r},$$

where OP is *any* path C from O to P (see Figure 13) and the minus sign is a conventional choice consistent with U being a potential energy when $\mathbf{F}$ is a force field. Then, for any two points A and B in the field,

$$\int_{AB} \mathbf{F} \cdot d\mathbf{r} = \int_{OB} \mathbf{F} \cdot d\mathbf{r} - \int_{OA} \mathbf{F} \cdot d\mathbf{r} = -\big(U(\mathbf{r}_B) - U(\mathbf{r}_A)\big), \qquad (31)$$

where $U(\mathbf{r}_A)$ and $U(\mathbf{r}_B)$ denote the values of U at the points A and B, respectively. The scalar function $U(\mathbf{r})$ is a scalar field called a **potential field** or simply a **potential** of the vector field $\mathbf{F}$.

When the vector field $\mathbf{F}$ is a force acting on a particle, the potential field U is the potential energy of the particle. Note that only *differences* of potential are defined by equation (31). As in one dimension, we can choose the value of the potential field to be zero at any convenient point (the *datum*). In this case, the datum is the origin O.

The potential energy of a particle was discussed in Unit 9.

Example 13

A potential of the two-dimensional field $\mathbf{F} = 2x\mathbf{i} - y\mathbf{j}$ is

$$U(\mathbf{r}) = U(x, y) = -(x^2 - \tfrac{1}{2}y^2).$$

Determine the line integral of $\mathbf{F}$ along a path from the origin to the point $(-3, 7)$.

Solution

Using equation (31), the line integral of $\mathbf{F}$ is given by

$$-\big(U(-3,7) - U(0,0)\big) = -\big(-((-3)^2 - \tfrac{1}{2} \times 7^2) - 0\big) = -\tfrac{31}{2}.$$

We can show that any scalar field U is a potential of the vector field $-\mathbf{grad}\, U$, by evaluating the line integral $\int_C (-\mathbf{grad}\, U) \cdot d\mathbf{r}$. We use the Cartesian form of $\mathbf{grad}\, U$,

$$\mathbf{grad}\, U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k},$$

in equation (25). Thus

$$\int_C (-\mathbf{grad}\, U) \cdot d\mathbf{r} = -\int_{t_0}^{t_1} \left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial z}\frac{dz}{dt} \right) dt.$$

The integrand is the expression for dU/dt as given by the chain rule. So we have

The chain rule was discussed in Unit 7.

$$\int_C (-\mathbf{grad}\, U) \cdot d\mathbf{r} = -\int_{t_0}^{t_1} \frac{dU}{dt}\, dt = -\big(U(t_1) - U(t_0)\big).$$

Notice again the abuse of notation. Since $\mathbf{r} = \mathbf{r}(t)$, $U(\mathbf{r})$ becomes $U(t)$.

Comparing this result with equation (31), you can see that the scalar field U is a potential of the vector field $\mathbf{F} = -\mathbf{grad}\, U$. If we now choose a datum at the origin, we have

$$\mathbf{F}(\mathbf{r}) = -\mathbf{grad}\, U \quad \text{and} \quad U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r}, \tag{32}$$

where $\mathbf{r}$ is the position vector of P, any point in space. These statements generalise, to two and three dimensions, the one-dimensional relationships

$$F(x) = -\frac{dU}{dx} \quad \text{and} \quad U(a) = -\int_0^a F(x)\, dx,$$

See Unit 9.

where a is any point on the x-axis.

Exercise 34

Given that the line integral of the field $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$ is path-independent, find a potential function $U(x, y, z)$ for $\mathbf{F}$.

The approach taken in the solution to Exercise 34 is summarised in the following procedure.

Procedure 1 Determining the potential function from the line integral of a force

Suppose that we are given the force $\mathbf{F}(\mathbf{r})$ whose line integral is path-independent. To determine the potential function $U(\mathbf{r})$, with the datum set at the origin, carry out the following steps.

1. Take C to be the direct path from $(0, 0, 0)$ to the general point (a, b, c) parametrised by
$$\mathbf{r} = at\mathbf{i} + bt\mathbf{j} + ct\mathbf{k} \quad (0 \leq t \leq 1).$$
2. With this choice of parametrisation, calculate the scalar line integral
$$U(a, b, c) = -\int_C \mathbf{F} \cdot d\mathbf{r} = -\int_0^1 \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt,$$
and write $U(\mathbf{r}) = U(x, y, z)$.

In Subsection 3.1 we considered the one-dimensional case of a force $F\mathbf{i}$ accelerating a particle along the x-axis, and showed that the work done by the force is equal to the change of kinetic energy of the particle. By combining equations (23) and (31), we can see that when F depends on position x, we are able to express the work done by the force as minus the change of potential energy. Furthermore, equation (21) shows that the work done equals the gain in kinetic energy. Thus for any force $F(x)\,\mathbf{i}$ that acts parallel to the x-axis and depends only on position x, the total mechanical energy (i.e. kinetic energy plus potential energy) is conserved.

Compare the discussion in Unit 9, Section 4.

We now have several links between line integrals of a force $\mathbf{F}$, path independence, the curl of $\mathbf{F}$, a potential for $\mathbf{F}$, and conservation of mechanical energy. In the next subsection we tie all these ideas together with various definitions of a conservative vector field.

4.2 Conservative vector fields

We classify vector fields according to whether their line integrals are path-dependent or path-independent. Those vector fields for which *all* the line integrals between *all* pairs of points are path-independent are called **conservative fields**. If there is at least one pair of points for which the line integrals of the vector field are path-dependent, then the vector field is *non-conservative*.

Conservative fields: Property (a)

The line integral of a conservative vector field along a path between any two fixed points A and B depends only on the starting point A and the endpoint B of the path. It is independent of the actual path taken between the two points.

Note that in one dimension, where the particle is restricted to the x-axis, the only possibility for varying the path between two fixed points $x = a$ and $x = b$ is when the particle overshoots into the regions $x > b$ or $x < a$ (see Figure 7). But then the particle always has to reverse back through these regions to reach $x = a$ or $x = b$, so the net contribution to the line integral from these forward and reverse motions vanishes. So all one-dimensional vector fields $F(x)\,\mathbf{i}$ that depend only on position x are conservative.

It follows from Property (a) that if you are asked to evaluate the line integral of a conservative field between any two given points, then you are free to choose the path that produces the simplest possible evaluation.

Example 14

Given that the vector field

$$\mathbf{F} = 2x\mathbf{i} - y\mathbf{j}$$

is a conservative field, evaluate the line integral of $\mathbf{F}$ from the origin to the point $(2, 0)$.

This field was considered in Example 13.

Solution

We are free to choose the path for a conservative field. The simplest path to take is the segment of the x-axis from $x = 0$ to $x = 2$. This path can be parametrised by

$$x = t, \quad y = 0 \quad (0 \leq t \leq 2),$$

so $dx/dt = 1$ and $dy/dt = 0$. The components of $\mathbf{F}$ on the path are $F_1 = 2x = 2t$ and $F_2 = -y = 0$, so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 2t\,dt = 4.$$

The line integral of a conservative field around any closed curve is zero. To see this, consider the loop in Figure 14. The two points A and B divide the loop into two segments, an upper segment APB and a lower segment AQB. We know that the line integral of any conservative field $\mathbf{F}$ from A to B has the same value for both segments, that is,

$$\int_{APB} \mathbf{F} \cdot d\mathbf{r} = \int_{AQB} \mathbf{F} \cdot d\mathbf{r}. \tag{33}$$

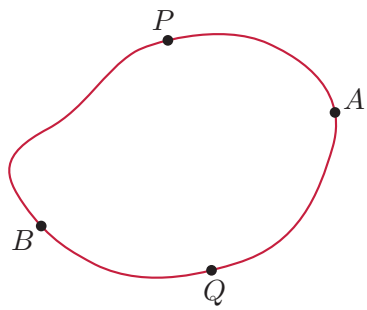


Figure 14 Closed path

Now the line integral for one complete anticlockwise traversal of the loop, starting and ending at A, consists of the line integral along APB plus that along BQA, that is,

$$\oint_{APBQA} \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} + \int_{BQA} \mathbf{F} \cdot d\mathbf{r}.$$

But we know from equation (28) that

$$\int_{BQA} \mathbf{F} \cdot d\mathbf{r} = -\int_{AQB} \mathbf{F} \cdot d\mathbf{r}.$$

Hence

$$\oint_{APBQA} \mathbf{F} \cdot d\mathbf{r} = \int_{APB} \mathbf{F} \cdot d\mathbf{r} - \int_{AQB} \mathbf{F} \cdot d\mathbf{r} = 0,$$

by equation (33). A similar argument applies to any other loop in the domain of $\mathbf{F}$.

This gives us another important property of a conservative field.

Conservative fields: Property (b)

The line integral of any conservative field $\mathbf{F}$ around *any closed curve* C is zero, that is,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0. \tag{34}$$

Note that the line integral of a conservative field must be zero for *all possible loops* in its domain. If just one loop has a non-zero line integral, that is sufficient to make the field non-conservative.

Exercise 35

State which of the two vector fields $\mathbf{u}$ and $\mathbf{v}$ in Exercise 29 is definitely non-conservative.

How can we tell whether or not a given vector field is conservative? To use Property (a), we should need to know that all the line integrals between all possible pairs of fixed points are path-independent. It is obviously not practicable to evaluate all possible line integrals. Alternatively, we could think of using Property (b), but this would seem to require us to evaluate the line integrals around all possible loops! What we need is a simple test for a conservative field. What about the result involving curl that we illustrated via equation (30)? Let us see how this relates to conservative fields. We know that all line integrals around closed loops are zero if $\mathbf{F}$ is a conservative field. Hence from equation (30) and the corresponding x- and y-components, $\mathbf{curl}\,\mathbf{F}$ must be zero for any conservative field.

Conservative fields: Property (c)

If $\mathbf{F}$ is a conservative field, then $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere.

Property (c) holds for all conservative fields, but it does not provide us with a sufficient test for conservative fields. Perhaps there are vector fields $\mathbf{F}$ with $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ that are non-conservative? However, it can be shown that the converse of Property (c) is also true *provided* that the domain of $\mathbf{F}$ is *simply-connected*. Hence we have the following 'curl test' to determine if a field is conservative.

Curl test for a conservative field

If $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ everywhere, then $\mathbf{F}$ is a conservative field provided that the domain of $\mathbf{F}$ is simply-connected.

Thus it turns out that a conservative field is always irrotational, but conversely, an irrotational field is not necessarily conservative unless its domain is simply-connected. We will not prove this converse statement, but we will show in an example (in Exercise 40) where the curl test fails due to the domain being non-simply-connected. For all other cases in this module, you may assume that the curl test is a sufficient test for a conservative field.

We can use the curl test to show that if $\mathbf{F} = -\mathbf{grad}\,U$ for some scalar field U, then $\mathbf{F}$ is a conservative field. This follows from Exercise 12(d), where you found that $\mathbf{curl}\,(\mathbf{grad}\,f) \equiv \mathbf{0}$ for any differentiable scalar field f.

Conservative fields: Property (d)

If a vector field is a gradient, then it is conservative. Conversely, any conservative vector field can be expressed as a gradient of a scalar field.

Using equations (32), we can express this last property in terms of potentials.

Conservative fields: Property (e)

For a conservative field $\mathbf{F}$ and fixed origin O (the datum), there exists a unique potential field defined by

$$U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r}$$

with $\mathbf{F} = -\mathbf{grad}\,U$, where $\overrightarrow{OP} = \mathbf{r}$.

For a different datum, the potential field will differ from U by a constant.

Property (e) tells us that we can determine line integrals of a conservative field simply by finding its corresponding potential field and using equation (31), as we did in Example 13. What is more, Property (e) combined with equation (21) shows that mechanical energy, being the sum of kinetic and potential energies, is conserved when a particle is acted on by a conservative force field.

Exercise 36

(a) Use the curl test to determine which of the following vector fields are conservative:

$$\mathbf{h} = x\mathbf{i} + y\mathbf{j},$$
$$\mathbf{u} = \omega(-y\mathbf{i} + x\mathbf{j}), \quad \text{where } \omega \text{ is a positive constant,}$$
$$\mathbf{F} = 2x\mathbf{i} + (xz - 2)\mathbf{j} + xy\mathbf{k}.$$

(b) Show that $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$ for $\mathbf{G} = kr^n\,\widehat{\mathbf{r}}$, where k and n are non-zero constants, $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = |\mathbf{r}|$ and $\widehat{\mathbf{r}} = \mathbf{r}/r$ $(r \neq 0)$.

The vector field $\mathbf{G} = kr^n\,\widehat{\mathbf{r}}$ (for any n or k) in Exercise 36(b) is directed radially towards or away from the origin depending on the sign of the constant k. Such fields are called **central fields**. In this case, the magnitude of the central field depends only on the distance from the origin r, that is, it is spherically symmetric, and a general expression for a force of this type is given by $\mathbf{F} = f(r)\,\widehat{\mathbf{r}}$ (as in Exercise 20).

The domain of $\mathbf{G}$ is simply-connected, even for $n < 0$, since although $\mathbf{G}$ cannot be defined at the origin when $n < 0$, we know that the space $\mathbb{R}^3$ excluding the origin is still simply-connected. You have found that $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$, so it follows that $\mathbf{G}$ is conservative for all n. Indeed, since we know from Exercise 20 that $\mathbf{F}$ is irrotational, it follows that all spherically symmetric central fields $\mathbf{F}$ are conservative. For example, the force of gravity near a large spherical object such as a planet or a star is always directed towards the centre of the sphere, with magnitude depending only on the distance to the centre of the sphere. Thus the force of gravity is a spherically symmetric central field and is therefore conservative. We can use the law of conservation of mechanical energy for conservative forces for any problem involving gravitational forces only – for example, the motion of planets around the Sun or the motion of artificial Earth satellites.

We can express the law of conservation of mechanical energy as

$$\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 = U(\mathbf{r}_A) - U(\mathbf{r}_B), \tag{35}$$

This equation is a combination of equations (31) and (21), and we have used $v_A^2 = \mathbf{v}_A \cdot \mathbf{v}_A$ and $v_B^2 = \mathbf{v}_B \cdot \mathbf{v}_B$.

where the left-hand side is the change in kinetic energy when the body moves from A to B, with position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, and the right-hand side is minus the change in potential energy. Equation (35) can be rewritten as

$$\tfrac{1}{2}mv_A^2 + U(\mathbf{r}_A) = \tfrac{1}{2}mv_B^2 + U(\mathbf{r}_B),$$

which tells us that the total mechanical energy,

$$E = \tfrac{1}{2}mv^2 + U(\mathbf{r}),$$

is a constant throughout the motion, provided that the force is conservative. This generalises the law of conservation of energy, given in Unit 9, Section 4, to motion in more than one dimension.

Exercise 37

The gravitational potential energy field of a body of mass m in the vicinity of the Earth is given by

$$U(\mathbf{r}) = mgR\left(1 - \frac{R}{|\mathbf{r}|}\right) \quad (|\mathbf{r}| \geq R),$$

where the position vector $\mathbf{r}$ of the body is measured from the centre of the Earth, g is the magnitude of the acceleration due to gravity at the Earth's surface, R is the Earth's radius, and the datum is on the Earth's surface. Derive an expression for the Earth's gravitational force $\mathbf{F}(\mathbf{r})$ on the body.

Example 15

An Earth satellite moves in an elliptical orbit with its nearest and furthest points from the Earth's surface being $\frac{1}{2}R$ and $2R$, where R is the Earth's radius. If its slowest speed is u, find its fastest speed. Assume that the only force acting on the satellite is the Earth's gravity.

Solution

Gravity is a conservative field, so mechanical energy is conserved throughout the motion of the satellite. The potential energy function U is given in Exercise 37. We can use the law of conservation of mechanical energy, equation (35), with A the furthest point reached by the satellite, $|\mathbf{r}_A| = 3R$, and B the nearest point, $|\mathbf{r}_B| = \frac{3}{2}R$, measured from the centre of the Earth. Then the speed at A is given as $v_A = u$, and we need to find v_B, the speed at B. Thus

$$\begin{aligned}\tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mu^2 &= U(\mathbf{r}_A) - U(\mathbf{r}_B)\\ &= mgR\left(\left(1 - \frac{R}{3R}\right) - \left(1 - \frac{2R}{3R}\right)\right)\\ &= \tfrac{1}{3}mgR.\end{aligned}$$

Hence the fastest speed is

$$v_B = \sqrt{u^2 + 2gR/3}.$$

All spherically symmetric central fields are conservative, but not all conservative fields are central fields. Consider, for example, the two-dimensional vector field $\mathbf{F}(\mathbf{r}) = -2px\mathbf{i} - py\mathbf{j}$, where p is a constant. The vector $\mathbf{F}(\mathbf{r})$ at any point is not, in general, parallel to the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ at that point, so the field is not a central field. However, this vector field is conservative, as you can confirm in the next exercise.

Exercise 38

The force acting on a particle is given by

$$\mathbf{F} = -2px\mathbf{i} - py\mathbf{j},$$

where p is a constant.

(a) Show that the force is a conservative field.

(b) Confirm that $U(x, y) = p(x^2 + \frac{1}{2}y^2)$ is a potential energy function for $\mathbf{F}$, and hence find the work done by the force, in terms of p, when the particle moves from $(5, 0)$ to $(0, 5)$.

Finally, it is useful to gather together the properties of conservative fields.

Properties of conservative fields

Let $\mathbf{F}$ be a conservative vector field.

(a) All line integrals of $\mathbf{F}$ between any two fixed points in the domain are path-independent.

(b) The line integrals of $\mathbf{F}$ around all closed curves in the domain are zero.

(c) The curl of $\mathbf{F}$ is zero everywhere in the domain.

(d) All gradient fields are conservative, and a conservative vector field can be expressed as the gradient of a scalar field.

(e) For a fixed origin O (the datum), there exists a unique potential field defined by

$$U(\mathbf{r}) = -\int_{OP} \mathbf{F} \cdot d\mathbf{r}$$

with $\mathbf{F} = -\mathbf{grad}\, U$, where $\overrightarrow{OP} = \mathbf{r}$.

We have taken statement (a) as the definition of a conservative field, but statements (b) and (e) are entirely equivalent to statement (a) and may alternatively be taken as the definition. Statement (c) is also equivalent to each of statements (a), (b) and (e) *provided* that the domain of $\mathbf{F}$ is simply-connected.

Exercise 39

Show that any vector field of the form $\mathbf{F} = f(x)\,\mathbf{i} + g(y)\,\mathbf{j}$, where f is a function of x only, and g is a function of y only, is a conservative field. Can a vector field of the form $\mathbf{H} = g(y)\,\mathbf{i} + f(x)\,\mathbf{j}$ be conservative?

Exercise 40

Consider the two-dimensional vector field

$$\mathbf{B}(x, y) = \frac{k(-y\mathbf{i} + x\mathbf{j})}{x^2 + y^2} \quad (x^2 + y^2 > 0),$$

where k is a constant.

(a) Show that

$$\oint_C \mathbf{B} \cdot d\mathbf{r} = 2\pi k,$$

where C is a circle of radius a with centre at the origin.

(b) Show that $\mathbf{curl}\,\mathbf{B} = \mathbf{0}$.

Learning outcomes

After studying this unit, you should be able to:

- calculate the divergence of a vector field
- calculate the curl of a vector field
- appreciate the use of gradient, divergence and curl to model simple laws and problems involving heat flow, fluid flow and local rotation in a vector field
- evaluate scalar line integrals of vector fields
- solve simple problems in one, two and three dimensions involving changes of kinetic energy and potential energy, and the work done by a force
- use the curl test for identifying conservative fields.

Solutions to exercises

Solution to Exercise 1

The components of $\mathbf{F}$ are

$$F_1 = xy, \quad F_2 = 0, \quad F_3 = -yz.$$

The relevant partial derivatives are

$$\frac{\partial F_1}{\partial x} = y, \quad \frac{\partial F_2}{\partial y} = 0, \quad \frac{\partial F_3}{\partial z} = -y.$$

Hence $\operatorname{div}\mathbf{F} = y + 0 + (-y) = 0$.

The scalar field $\operatorname{div}\mathbf{F}$ is zero everywhere, so $\operatorname{div}\mathbf{F}(3,-1,2) = 0$.

Solution to Exercise 2

The components of $\mathbf{F}$ are $F_1 = xy$, $F_2 = yz$ and $F_3 = zx$, so

$$\frac{\partial F_1}{\partial x} = y, \quad \frac{\partial F_2}{\partial y} = z, \quad \frac{\partial F_3}{\partial z} = x.$$

Thus $\boldsymbol{\nabla}\cdot\mathbf{F} = y + z + x$ and $\boldsymbol{\nabla}\cdot\mathbf{F}(1,2,3) = 2 + 3 + 1 = 6$.

Solution to Exercise 3

The position vector in Cartesian coordinates is $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so

$$\widehat{\mathbf{r}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}},$$

$$\mathbf{F}(x,y,z) = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{(n+1)/2}}.$$

The x-component is

$$F_1 = \frac{x}{(x^2 + y^2 + z^2)^{(n+1)/2}},$$

and

$$\frac{\partial F_1}{\partial x} = \frac{(x^2 + y^2 + z^2)^{(n+1)/2} - x^2(n+1)(x^2 + y^2 + z^2)^{(n-1)/2}}{(x^2 + y^2 + z^2)^{n+1}}.$$

The component F_2 and the partial derivative $\partial F_2/\partial y$ are obtained from F_1 and $\partial F_1/\partial x$ by interchanging x and y. Similarly, F_3 and $\partial F_3/\partial z$ are obtained by interchanging x and z. When this is done, we obtain

$$\begin{aligned}\operatorname{div}\mathbf{F} &= \frac{3(x^2 + y^2 + z^2)^{(n+1)/2} - (x^2 + y^2 + z^2)(n+1)(x^2 + y^2 + z^2)^{(n-1)/2}}{(x^2 + y^2 + z^2)^{n+1}} \\ &= \frac{(3 - (n+1))(x^2 + y^2 + z^2)^{(n+1)/2}}{(x^2 + y^2 + z^2)^{n+1}} \\ &= \frac{2 - n}{(x^2 + y^2 + z^2)^{(n+1)/2}}.\end{aligned}$$

Hence $\operatorname{div}\mathbf{F} = 0$ everywhere only when $n = 2$.

Solution to Exercise 4

We have

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j},$$

so

$$\frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\cos\phi\,\mathbf{i} - \sin\phi\,\mathbf{j} = -\mathbf{e}_\rho.$$

Solution to Exercise 5

In spherical coordinates, $\mathbf{r} = r\mathbf{e}_r$. Hence, using equation (3),

$$\operatorname{div}\mathbf{r} = \frac{\partial r}{\partial r} + \frac{2r}{r} = 3.$$

This should be compared to the result given in Example 1(a) for $\operatorname{div}\mathbf{r}$, which was obtained using Cartesian coordinates.

Solution to Exercise 6

The heat flow vector field $\mathbf{J}$ has cylindrical symmetry. Anywhere on the outer surface, $\mathbf{J}$ is directed radially outwards and has magnitude

$$|\mathbf{J}| = A\sqrt{x^2+y^2} = 2\times 10^6 \times 0.02 = 4\times 10^4.$$

The total surface area (in m^2) of a one-metre length of rod is

$$2\pi \times 0.02 \times 1 = 4\pi \times 10^{-2}.$$

Hence the total outward heat flow rate from a one-metre length is given (in watts) by

$$(4\times 10^4) \times (4\pi \times 10^{-2}) = 1600\pi,$$

that is, the heat flow rate is about 5 kW.

This is also the rate of heat generated in the one-metre length, since the heat flow is steady. Alternatively,

$$\begin{aligned} S \times \text{volume} &= (4\times 10^6) \times \pi(0.02)^2 \times 1 \\ &= 1600\pi. \end{aligned}$$

Solution to Exercise 7

In cylindrical coordinates, $\rho = \sqrt{x^2+y^2}$ and $\mathbf{e}_\rho = (x\mathbf{i}+y\mathbf{j})/\sqrt{x^2+y^2}$. Hence $\mathbf{J}(\rho,\phi,z) = A\rho\,\mathbf{e}_\rho$, and the cylindrical components of $\mathbf{J}$ are $J_\rho = A\rho$, $J_\phi = J_z = 0$. The only non-zero partial derivative is $\partial J_\rho/\partial\rho = A$. Hence, using equation (2), we have

$$\operatorname{div}\mathbf{J} = A + \frac{1}{\rho}A\rho = 2A,$$

which is in agreement with the result of Example 3.

Solution to Exercise 8

(a) The components of $\mathbf{N}$ are (in $\mathrm{W\,m^{-2}}$)

$$N_1 = N_2 = 0, \quad N_3 = 5000,$$

so

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_2}{\partial y} = \frac{\partial N_3}{\partial z} = 0.$$

Thus $\operatorname{div}\mathbf{N} = 0$.

(b) The only non-zero component of $\mathbf{N}$ is $N_3 = 5000e^{-\alpha z}$, so

$$\frac{\partial N_1}{\partial x} = \frac{\partial N_2}{\partial y} = 0, \quad \frac{\partial N_3}{\partial z} = 5000(-\alpha)e^{-\alpha z}.$$

Hence $\operatorname{div}\mathbf{N} = -5000\alpha e^{-\alpha z}$.

Solution to Exercise 9

The only non-zero component of $\mathbf{f}$ is $f_\rho = \rho^{-n}$, so $\partial f_\rho/\partial\rho = -n\rho^{-(n+1)}$. From equation (2), we obtain

$$\operatorname{div}\mathbf{f} = \frac{-n}{\rho^{n+1}} + \frac{1}{\rho}\frac{1}{\rho^n} = \frac{1-n}{\rho^{n+1}}.$$

This is zero everywhere only when $n = 1$.

Solution to Exercise 10

We have

$$\operatorname{div}\mathbf{F} = 0 + 0 + 0 = 0,$$
$$\operatorname{div}\mathbf{G} = 2x + 2y + 2z,$$
$$\operatorname{div}\mathbf{H} = 1 + 1 - 2 = 0.$$

So $\mathbf{F}$ and $\mathbf{H}$ could be magnetic fields, but $\mathbf{G}$ could not, since the divergence of a magnetic field is always zero.

Solution to Exercise 11

(a) The components of $\mathbf{F}(x, y, z) = x(y - z)\mathbf{i} + 3x^2\mathbf{j} + yz\mathbf{k}$ are $F_1 = x(y - z)$, $F_2 = 3x^2$ and $F_3 = yz$.

The partial derivatives required for substitution in equation (14) are

$$\frac{\partial F_1}{\partial y} = x, \quad \frac{\partial F_1}{\partial z} = -x,$$
$$\frac{\partial F_2}{\partial x} = 6x, \quad \frac{\partial F_2}{\partial z} = 0,$$
$$\frac{\partial F_3}{\partial x} = 0, \quad \frac{\partial F_3}{\partial y} = z.$$

Hence, using equation (14),

$$\mathbf{curl}\,\mathbf{F} = z\mathbf{i} - x\mathbf{j} + 5x\mathbf{k}.$$

(b) (i) At the origin, $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$.

(ii) At $(1, 2, 3)$, $\mathbf{curl}\,\mathbf{F} = 3\mathbf{i} - \mathbf{j} + 5\mathbf{k}$.

(iii) At $(0, 0, 5)$, $\mathbf{curl}\,\mathbf{F} = 5\mathbf{i}$.

Solution to Exercise 12

(a) The components of $\mathbf{F} = \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ are

$$F_1 = x, \quad F_2 = y, \quad F_3 = z,$$

so

$$\frac{\partial F_1}{\partial z} = \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial z} = \frac{\partial F_2}{\partial x} = \frac{\partial F_3}{\partial y} = \frac{\partial F_3}{\partial x} = 0.$$

Hence, substituting into equation (14),

$$\mathbf{curl}\,\mathbf{F} = \mathbf{0}.$$

(b) The components of $\mathbf{F} = f(r)\,\widehat{\mathbf{r}} = (f(r)/r)\mathbf{r}$ are

$$F_1 = f(r)\,\frac{x}{r}, \quad F_2 = f(r)\,\frac{y}{r}, \quad F_3 = f(r)\,\frac{z}{r}.$$

The x-component of $\mathbf{curl}\,\mathbf{F}$ (using the chain rule) is

$$\begin{aligned}\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= z\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial y} - y\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{\partial r}{\partial z} \\ &= z\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{y}{r} - y\,\frac{d}{dr}\left(\frac{f(r)}{r}\right)\frac{z}{r} \\ &= 0.\end{aligned}$$

Note that we have used $\partial r/\partial y = y(x^2 + y^2 + z^2)^{-1/2} = y/r$ and $\partial r/\partial z = z/r$. Similarly, the y- and z-components of $\mathbf{curl}\,\mathbf{F}$ are also zero, so

$$\mathbf{curl}\,(f(r)\,\widehat{\mathbf{r}}) = \mathbf{0}$$

for all functions f of r only.

(c) We have

$$\mathbf{F} = (y^2 + 2z)\mathbf{i} + (xy + 6z)\mathbf{j} + (z^2 + 2xz + y)\mathbf{k},$$

so the x-, y- and z-components of $\mathbf{curl}\,\mathbf{F}$ are

$$\begin{aligned}\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} &= 1 - 6 = -5, \\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} &= 2 - 2z = 2(1 - z), \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= y - 2y = -y,\end{aligned}$$

thus

$$\mathbf{curl}\,\mathbf{F} = -5\mathbf{i} + 2(1 - z)\mathbf{j} - y\mathbf{k}.$$

(d) Since

$$\mathbf{F} = \mathbf{grad}\, f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

we have

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\,\partial z} - \frac{\partial^2 f}{\partial z\,\partial y} = 0.$$

Similarly, the y- and z-components are zero. Hence

$$\mathbf{curl}\,(\mathbf{grad}\, f) \equiv \mathbf{0}.$$

Solution to Exercise 13

With $\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}$ and $\mathbf{B} = B_1\mathbf{i} + B_2\mathbf{j} + B_3\mathbf{k}$, using equation (14) to determine $\mathbf{curl}\,\mathbf{A} = \mathbf{B}$, we find

$$B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}, \quad B_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}, \quad B_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

Hence

$$\begin{aligned}\operatorname{div}\mathbf{B} &= \frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \\ &= \frac{\partial^2 A_3}{\partial x\,\partial y} - \frac{\partial^2 A_2}{\partial x\,\partial z} + \frac{\partial^2 A_1}{\partial y\,\partial z} - \frac{\partial^2 A_3}{\partial y\,\partial x} + \frac{\partial^2 A_2}{\partial z\,\partial x} - \frac{\partial^2 A_1}{\partial z\,\partial y} \\ &= 0,\end{aligned}$$

where we have used the mixed derivative theorem for second-order partial derivatives (Unit 7, Subsection 2.2), which assumes that $\mathbf{A}$ (and therefore A_1, A_2 and A_3) is sufficiently smooth.

Solution to Exercise 14

Any vector field having zero curl everywhere is an irrotational field. The fields in parts (a), (b) and (d) of Exercise 12 and part (a)(i) of Example 4 are irrotational.

Solution to Exercise 15

The field $\mathbf{F}$ is confined to the (x, y)-plane, so

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial}{\partial x}(g(y)) - \frac{\partial}{\partial y}(f(x))\right)\mathbf{k} = \mathbf{0}.$$

Solution to Exercise 16

Both $\mathbf{u}$ and $\mathbf{v}$ are two-dimensional fields in the (x, y)-plane.

For the field $\mathbf{u}$,

$$u_1 = -\omega y \quad \text{and} \quad u_2 = \omega x,$$

so

$$\frac{\partial u_2}{\partial x} = \omega, \quad \frac{\partial u_1}{\partial y} = -\omega.$$

Hence

$$\mathbf{curl}\,\mathbf{u} = \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\mathbf{k} = 2\omega\mathbf{k}.$$

For the field $\mathbf{v}$,

$$v_1 = -\frac{\omega a^2 y}{x^2+y^2} \quad \text{and} \quad v_2 = \frac{\omega a^2 x}{x^2+y^2},$$

so

$$\frac{\partial v_2}{\partial x} = \frac{(x^2+y^2)\omega a^2 - \omega a^2 x(2x)}{(x^2+y^2)^2} = \frac{\omega a^2(y^2-x^2)}{(x^2+y^2)^2},$$
$$\frac{\partial v_1}{\partial y} = \frac{(x^2+y^2)(-\omega a^2) + \omega a^2 y(2y)}{(x^2+y^2)^2} = \frac{\omega a^2(y^2-x^2)}{(x^2+y^2)^2}.$$

Hence

$$\mathbf{curl}\,\mathbf{v} = \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Note that at the boundary of the central region, $\mathbf{curl}\,\mathbf{u} \neq \mathbf{curl}\,\mathbf{v}$.

Solution to Exercise 17

In cylindrical coordinates we have

$$x = \rho\cos\phi, \quad y = \rho\sin\phi.$$

Hence

$$\mathbf{u}(\rho,\phi,z) = -\rho\omega\sin\phi\,\mathbf{i} + \rho\omega\cos\phi\,\mathbf{j} = \omega\rho\,\mathbf{e}_\phi \quad (\rho \leq a),$$

and since $x^2+y^2=\rho^2$,

$$\mathbf{v}(\rho,\phi,z) = \frac{a^2}{\rho^2}\omega\rho\,\mathbf{e}_\phi = \frac{\omega a^2}{\rho}\,\mathbf{e}_\phi \quad (\rho \geq a).$$

The vector field lines of $\mathbf{u}$ and $\mathbf{v}$ are circles.

Solution to Exercise 18

(a) Using a Cartesian coordinate system with the z-axis lying on the axis of the Earth's rotation and in the direction of the angular velocity, we have $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Hence

$$\mathbf{v}(x,y,z) = \omega\mathbf{k} \times (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = \omega(x\mathbf{j} - y\mathbf{i}).$$

(b) Now $v_1 = -\omega y$, $v_2 = \omega x$ and $v_3 = 0$, so using equation (15) we have

$$\mathbf{curl}\,\mathbf{v} = \left(\frac{\partial}{\partial x}(\omega x) - \frac{\partial}{\partial y}(-\omega y)\right)\mathbf{k} = 2\omega\mathbf{k} = 2\boldsymbol{\omega}.$$

Hence

$$|\mathbf{curl}\,\mathbf{v}| = 2\omega = 2 \times \frac{2\pi}{24\times 60\times 60} = 1.454\times 10^{-4}.$$

(c) Every object fixed on the Earth (including your home) rotates with angular velocity $\boldsymbol{\omega}$. So from part (b), the rate of rotation (in $\mathrm{rad\,s^{-1}}$) is

$$|\boldsymbol{\omega}| = \tfrac{1}{2}|\mathbf{curl}\,\mathbf{v}| = 7.272\times 10^{-5}.$$

Solution to Exercise 19

Note that $\mathbf{F}$ has no z-component and no z-dependence in the other components, so it is a two-dimensional vector field. Hence we can use equation (18) to determine $\mathbf{curl}\,\mathbf{F}$, from which, with $F_\rho = f(\rho)$ and $F_\phi = \rho^n g(\phi)$, we have

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \left(\frac{\partial F_\phi}{\partial \rho} + \frac{F_\phi}{\rho} - \frac{1}{\rho}\frac{\partial F_\rho}{\partial \phi}\right)\mathbf{e}_z \\ &= \left(n\rho^{n-1}g(\phi) + \frac{\rho^n g(\phi)}{\rho} + 0\right)\mathbf{e}_z \\ &= (n+1)\rho^{n-1}g(\phi)\,\mathbf{e}_z.\end{aligned}$$

Since in general $g(\phi) \neq 0$, we have $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ (i.e. $\mathbf{F}$ is irrotational) only when $n = -1$.

Solution to Exercise 20

In spherical coordinates, $\mathbf{F} = f(r)\,\mathbf{e}_r$ (since $\widehat{\mathbf{r}} = \mathbf{e}_r$), so $F_r = f(r)$, $F_\theta = F_\phi = 0$ and, since F_r does not depend on θ and ϕ, $\partial F_r/\partial\theta = \partial F_r/\partial\phi = 0$. Thus, from equation (17), it immediately follows that $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$ and $\mathbf{F}$ is irrotational.

Solution to Exercise 21

We have $A_r = A_\theta = 0$ and $A_\phi = \psi(r,\theta)/(r\sin\theta)$. Writing $\mathbf{F} = F_r\mathbf{e}_r + F_\theta\mathbf{e}_\theta + F_\phi\mathbf{e}_\phi$ and using equation (17) for $\mathbf{F} = \mathbf{curl}\,\mathbf{A}$, for each spherical component in $\mathbf{F}$ we find

$$\begin{aligned}F_r &= \frac{1}{r}\frac{\partial A_\phi}{\partial\theta} + \frac{\cot\theta}{r}A_\phi \\ &= \frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\psi}{\sin\theta}\right) + \frac{\psi\cot\theta}{r^2\sin\theta} \\ &= \frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta} - \psi\cos\theta + \psi\cos\theta\right) = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad (36)\end{aligned}$$

$$\begin{aligned}F_\theta &= -\left(\frac{\partial A_\phi}{\partial r} + \frac{1}{r}A_\phi\right) \\ &= -\frac{1}{\sin\theta}\left(\frac{\partial}{\partial r}\left(\frac{\psi}{r}\right) + \frac{\psi}{r^2}\right) \\ &= -\frac{1}{r^2\sin\theta}\left(r\frac{\partial\psi}{\partial r} - \psi + \psi\right) = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \qquad (37)\end{aligned}$$

$$F_\phi = 0.$$

Since $\mathbf{F} = \mathbf{e}_r/r^2$, we require $F_r = 1/r^2$, which implies from equation (36) that $\partial\psi/\partial\theta = \sin\theta$. Also, we require $F_\theta = 0$, which implies from equation (37) that $\partial\psi/\partial r = 0$. Thus we have the required partial derivatives

$$\frac{\partial\psi}{\partial r} = 0, \quad \frac{\partial\psi}{\partial\theta} = \sin\theta.$$

The first of these partial derivatives implies that $\psi(r,\theta)$ is a function of θ only, which we write as $\psi(r,\theta) = f(\theta)$. The second partial derivative then implies that $f'(\theta) = \sin\theta$, which is integrated to give $f(\theta) = -\cos\theta + c$, where c is an arbitrary constant. Hence the required expression for $\psi(r,\theta)$.

Note that using the result found in Exercise 13, since $\mathbf{F} = \mathbf{curl}\,\mathbf{A}$, it immediately follows that $\operatorname{div}\mathbf{F} = 0$ everywhere (in the domain of $\mathbf{A}$), which is consistent with the findings of Exercise 3 and Example 2 (for $n = 2$).

Solution to Exercise 22

The work done is

$$W = -A\int_3^1 \frac{1}{x^2}\,dx = -A\left[-\frac{1}{x}\right]_3^1 = A\left(1-\tfrac{1}{3}\right) = \tfrac{2}{3}A.$$

The change in kinetic energy is $\frac{2}{3}A$, so $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{2}{3}A$ and

$$v = \left(\tfrac{4}{3}(A/m) + u^2\right)^{1/2}.$$

Solution to Exercise 23

We have $dx/dt = -2\sin t$, $dy/dt = 2\cos t$, $F_1 = 2x + y = 4\cos t + 2\sin t$, $F_2 = -x = -2\cos t$. Hence

$$\begin{aligned}\int_{C_1}\mathbf{F}\cdot d\mathbf{r} &= \int_0^{\pi/2}\big((4\cos t + 2\sin t)(-2\sin t) + (-2\cos t)2\cos t\big)\,dt\\ &= \int_0^{\pi/2}\big(-8\sin t\cos t - 4(\sin^2 t + \cos^2)\big)\,dt\\ &= \int_0^{\pi/2}(-4\sin(2t) - 4)\,dt\\ &= \big[2\cos(2t) - 4t\big]_0^{\pi/2} = -4 - 2\pi.\end{aligned}$$

Note that for the third equality, we used the trigonometric identities $2\sin t\cos t = \sin 2t$ and $\sin^2 t + \cos^2 t = 1$.

Solution to Exercise 24

Put $x = t$. Then $y = 1 - t^2$, with t going from 1 to 0. The components of the vector field $\mathbf{F}$ are $F_1 = x^2 = t^2$ and $F_2 = xy = t(1-t^2)$. The derivatives are $dx/dt = 1$ and $dy/dt = -2t$. Hence

$$\begin{aligned}\int_C\mathbf{F}\cdot d\mathbf{r} &= \int_1^0\big(t^2 - 2t^2(1-t^2)\big)\,dt\\ &= \int_1^0(2t^4 - t^2)\,dt\\ &= \big[\tfrac{2}{5}t^5 - \tfrac{1}{3}t^3\big]_1^0 = -\left(\frac{6-5}{15}\right) = -\tfrac{1}{15}.\end{aligned}$$

Solution to Exercise 25

There are infinitely many paths between the points $(0,0,0)$ and $(1,1,1)$, so we choose two examples. The first moves from the origin along the x-axis to $(1,0,0)$, then parallel to the y-axis to $(1,1,0)$, and finally parallel to the z-axis to $(1,1,1)$. We can parametrise this as

$$\mathbf{r}(t) = \begin{cases} t\mathbf{i}, & 0 \le t \le 1, \\ \mathbf{i} + (t-1)\mathbf{j}, & 1 \le t \le 2, \\ \mathbf{i} + \mathbf{j} + (t-2)\mathbf{k}, & 2 \le t \le 3. \end{cases}$$

Another alternative is the straight line joining the two points, which can be parametrised much more easily, as

$$\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k} \quad (0 \le t \le 1).$$

Solution to Exercise 26

With $x = 2(1-t)$, $y = 2t$ and $z = 0$, we have $dx/dt = -2$, $dy/dt = 2$, $dz/dt = 0$, and $F_1 = 2x + y = 4 - 2t$, $F_2 = -x = -2(1-t)$. Hence

$$\begin{aligned} \int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \big((4-2t)(-2) - 2(1-t)2\big)\, dt \\ &= \int_0^1 (8t - 12)\, dt = \big[4t^2 - 12t\big]_0^1 = -8, \end{aligned}$$

which is the same value as that found in Example 9.

Solution to Exercise 27

In Exercise 23, C_1 is parametrised by $x = 2\cos t$, $y = 2\sin t$. Thus $dx/dt = -2\sin t$, $dy/dt = 2\cos t$. Also, $G_1 = x^2 = 4\cos^2 t$, $G_2 = y = 2\sin t$. Hence

$$\begin{aligned} \int_{C_1} \mathbf{G} \cdot d\mathbf{r} &= \int_0^{\pi/2} \big((4\cos^2 t)(-2\sin t) + 2\sin t(2\cos t)\big)\, dt \\ &= \int_0^{\pi/2} \big(-8\cos^2 t \sin t + 2\sin(2t)\big)\, dt \\ &= \big[\tfrac{8}{3}\cos^3 t - \cos(2t)\big]_0^{\pi/2} = -\tfrac{2}{3}. \end{aligned}$$

In Example 9, C_2 is parametrised by $x = t$, $y = 2 - t$. Thus $dx/dt = 1$, $dy/dt = -1$. Also, $G_1 = x^2 = t^2$, $G_2 = y = 2 - t$. Hence

$$\int_{C_2} \mathbf{G} \cdot d\mathbf{r} = \int_2^0 \big(t^2 + (2-t)(-1)\big)\, dt = \big[\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 - 2t\big]_2^0 = -\tfrac{2}{3}.$$

The closed curve $APBQA$ is the path C_1 plus the reverse of path C_2. Hence the scalar line integral of $\mathbf{G}$ along the closed curve $APBQA$ is $-\frac{2}{3} - (-\frac{2}{3}) = 0$.

Solution to Exercise 28

We have

$$x = t, \quad y = t^2, \quad z = t^3, \quad t_0 = 0, \quad t_1 = 1,$$
$$F_1 = 2t, \quad F_2 = t^4 - 2, \quad F_3 = t^3,$$
$$dx/dt = 1, \quad dy/dt = 2t, \quad dz/dt = 3t^2,$$

so

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 \big(2t + (t^4 - 2)(2t) + t^3(3t^2)\big)\, dt \\ &= \int_0^1 (5t^5 - 2t)\, dt \\ &= -\tfrac{1}{6}.\end{aligned}$$

Solution to Exercise 29

(a) $$\oint_C \mathbf{u} \cdot d\mathbf{r} = \int_0^{2\pi} \big((a\cos t/a^2)(-a\sin t) + (a\sin t/a^2)(a\cos t)\big)\, dt = 0.$$

(b) $$\begin{aligned}\oint_C \mathbf{v} \cdot d\mathbf{r} &= \omega \int_0^{2\pi} \big((-a\sin t)(-a\sin t) + (a\cos t)(a\cos t)\big)\, dt \\ &= \omega a^2 \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt \\ &= \omega a^2 \int_0^{2\pi} 1\, dt \\ &= 2\omega\pi a^2.\end{aligned}$$

Solution to Exercise 30

The x-axis is the line $y = 0$, so the vector field $\mathbf{v}$ on the x-axis is $\mathbf{v}(x, 0) = \omega x\mathbf{j}$, which has no component along the x-axis. Thus the tangential component of $\mathbf{v}$ along the line is zero. Hence the scalar line integral along the line segment on the x-axis is zero.

Solution to Exercise 31

(a) In these examples, the component of the vector field $\mathbf{F}$ in the direction tangential to the curve is constant.

Here the curve C is a segment of the x-axis, on which $\mathbf{i} \cdot \mathbf{F} = 0$, so the line integral is zero.

(b) Here the curve C is a segment of the z-axis, on which $\mathbf{k} \cdot \mathbf{F} = 5$, a constant, so the line integral is $5 \times (6 - 0) = 30$.

(c) The tangential component of $\mathbf{F}$ on the semicircle is $\mathbf{e}_\theta \cdot \mathbf{F}(3, \theta) = 3^2 = 9$, a constant, so the line integral is $9 \times 3\pi = 27\pi$.

Solution to Exercise 32

We have

$$\mathbf{r}(t) = t\,\mathbf{i} - \ln(\cos t)\,\mathbf{j},$$

so

$$\dot{\mathbf{r}}(t) = \mathbf{i} + \tan t\,\mathbf{j}.$$

Thus the required length is given by

$$\begin{aligned}\int_0^{\pi/4} \sqrt{1+\tan^2 t}\,dt &= \int_0^{\pi/4} \sec t\,dt \\ &= \left[\ln(\sec t + \tan t)\right]_0^{\pi/4} \\ &= \ln(\sqrt{2}+1) - \ln(1) \\ &= \ln(\sqrt{2}+1).\end{aligned}$$

In practice, although equation (29) is a very useful formula, the actual computations can involve complicated integrals.

Solution to Exercise 33

(a) The line integral was found in the solution to Exercise 29(b) to have the value $2\omega\pi a^2$. The area of the circle is πa^2, so the required limit is $2\omega\pi a^2/\pi a^2 = 2\omega$.

(b) From equation (14), $\mathbf{curl}\,\mathbf{v} = (\partial(\omega x)/\partial x - \partial(-\omega y)/\partial y)\mathbf{k} = 2\omega\mathbf{k}$, and $\mathbf{k}\cdot\mathbf{curl}\,\mathbf{v} = 2\omega$.

Solution to Exercise 34

We have $\mathbf{F} = 2xyz\mathbf{i} + x^2z\mathbf{j} + x^2y\mathbf{k}$. Since the line integral is path-independent, take the direct path from $(0,0,0)$ to a general point (a,b,c) parametrised by

$$\mathbf{r}(t) = ta\mathbf{i} + tb\mathbf{j} + tc\mathbf{k} \quad (0 \le t \le 1),$$

so

$$\frac{d\mathbf{r}}{dt} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}.$$

Then

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 (2atbtct\mathbf{i} + (at)^2ct\mathbf{j} + (at)^2bt\mathbf{k})\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_0^1 (2a^2bc + a^2bc + a^2bc)t^3\,dt \\ &= 4a^2bc\left[\tfrac{1}{4}t^4\right]_0^1 \\ &= a^2bc \\ &= U(0,0,0) - U(a,b,c).\end{aligned}$$

Setting the datum for the potential energy function at the origin, so that $U(0,0,0) = 0$, we can deduce that a potential function for $\mathbf{F}$ is

$$U(x,y,z) = -x^2yz.$$

Solution to Exercise 35

The field $\mathbf{v}$ is non-conservative, because its line integral around the closed loop C is non-zero.

We cannot yet say from the solution to Exercise 29 that the field $\mathbf{u}$ is conservative, since there may be loops on which the line integral of $\mathbf{u}$ is non-zero. (In fact, $\mathbf{u}$ *is* conservative.)

Solution to Exercise 36

(a) $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ everywhere, $\mathbf{curl}\,\mathbf{u} = 2\omega\mathbf{k}$, and

$$\begin{aligned}\mathbf{curl}\,\mathbf{F} &= \left(\frac{\partial(xy)}{\partial y} - \frac{\partial(xz-2)}{\partial z}\right)\mathbf{i} + \left(\frac{\partial(2x)}{\partial z} - \frac{\partial(xy)}{\partial x}\right)\mathbf{j} \\ &\quad + \left(\frac{\partial(xz-2)}{\partial x} - \frac{\partial(2x)}{\partial y}\right)\mathbf{k} \\ &= -y\mathbf{j} + z\mathbf{k}.\end{aligned}$$

Thus field $\mathbf{h}$ is conservative, since $\mathbf{curl}\,\mathbf{h} = \mathbf{0}$ everywhere, and the other two fields are non-conservative, since $\mathbf{curl}\,\mathbf{u} \neq \mathbf{0}$ and, in general, $\mathbf{curl}\,\mathbf{F} \neq \mathbf{0}$.

(b) This is just a special case of Exercise 20, where it was shown that a vector field of this type is irrotational. In this case $f(r)$ (in the notation of Exercise 20) is now given by $f(r) = kr^n$, and just as in Exercise 20, the simplest way to determine $\mathbf{curl}\,\mathbf{G}$ is to use the expression for curl in spherical coordinates after noting that $\widehat{\mathbf{r}} = \mathbf{e}_r$. Since $G_r = kr^n$ and $G_\theta = G_\phi = 0$, we have $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$.

Alternatively, we can use Cartesian coordinates as follows. Since $\mathbf{G} = kr^{n-1}\mathbf{r}$, we have $G_1 = kr^{n-1}x$, $G_2 = kr^{n-1}y$ and $G_3 = kr^{n-1}z$. The $\mathbf{i}$-component of $\mathbf{curl}\,\mathbf{G}$ is then

$$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = k(n-1)(zr^{n-2}y/r - yr^{n-2}z/r) = 0,$$

where we have used $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$. Similarly, the $\mathbf{j}$- and $\mathbf{k}$-components are zero, so $\mathbf{curl}\,\mathbf{G} = \mathbf{0}$.

Note that it is much simpler to use spherical coordinates in this case, where we have been able to exploit the spherical symmetry of the field.

Solution to Exercise 37

Express $U(\mathbf{r})$ in Cartesian or spherical coordinates, and use $\mathbf{F} = -\mathbf{grad}\,U$ (the coordinate forms of $\mathbf{grad}\,U$ are given in Unit 15). It is easiest to use spherical coordinates, since U depends only on the distance $r = |\mathbf{r}|$. (It is a spherically symmetric scalar field.) Thus we have

$$U(r,\theta,\phi) = mgR\left(1 - \frac{R}{r}\right)$$

and

$$\mathbf{grad}\,U = \frac{\partial U}{\partial r}\,\mathbf{e}_r = \frac{mgR^2}{r^2}\,\mathbf{e}_r.$$

Thus the gravitational force in spherical coordinates is

$$\mathbf{F}(r,\theta,\phi) = -\mathbf{grad}\, U = -\frac{mgR^2}{r^2}\,\mathbf{e}_r,$$

or in coordinate-free form,

$$\mathbf{F}(\mathbf{r}) = -\frac{mgR^2}{|\mathbf{r}|^2}\,\widehat{\mathbf{r}}.$$

Note that on the surface of the Earth, $\mathbf{F} = -mg\,\widehat{\mathbf{r}}$, as required.

Solution to Exercise 38

(a) The field $\mathbf{F}$ is confined to the (x,y)-plane, so

$$\mathbf{curl}\,\mathbf{F} = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} = \left(\frac{\partial(-py)}{\partial x} - \frac{\partial(-2px)}{\partial y}\right)\mathbf{k} = \mathbf{0}.$$

Hence $\mathbf{F}$ is conservative.

(b) We have

$$\mathbf{grad}\, U = \frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} = 2px\mathbf{i} + py\mathbf{j}.$$

Hence $\mathbf{F} = -\mathbf{grad}\, U$, so U is a potential energy function for $\mathbf{F}$.

The work done by the force is the line integral of $\mathbf{F}$ for any path from $(5,0)$ to $(0,5)$. From equation (31), this line integral is simply the difference between potential energies,

$$-(U(0,5) - U(5,0)) = -\tfrac{25}{2}p + 25p = \tfrac{25}{2}p.$$

Solution to Exercise 39

Use the curl test. We have $\mathbf{curl}\,\mathbf{F} = [\partial g(y)/\partial x - \partial f(x)/\partial y]\mathbf{k}$, where $\partial g/\partial x = \partial f/\partial y = 0$. Hence $\mathbf{curl}\,\mathbf{F} = \mathbf{0}$, so $\mathbf{F}$ is conservative.

We have $\mathbf{curl}\,\mathbf{H} = [\partial f(x)/\partial x - \partial g(y)/\partial y]\mathbf{k} = (f'(x) - g'(y))\mathbf{k}$. This can be zero, making $\mathbf{H}$ conservative, only if $f'(x) = g'(y)$, and since the left-hand side of this equation is a function of x only, and the right-hand side is a function of y only, both sides must equal the same constant, m say. Thus for $\mathbf{H}$ to be conservative, we must have $f(x) = mx + c_1$ and $g(y) = my + c_2$, for constants m, c_1 and c_2. Otherwise the curl is non-zero and $\mathbf{H}$ is non-conservative.

Solution to Exercise 40

(a) The circle C can be parametrised by $x = a\cos t$, $y = a\sin t$. Hence $B_1 = -(k/a)\sin t$, $B_2 = (k/a)\cos t$, $dx/dt = -a\sin t$, $dy/dt = a\cos t$. So

$$\oint_C \mathbf{B}\cdot d\mathbf{r} = \int_0^{2\pi} k\,dt = 2\pi k.$$

Alternatively, the field can be expressed in polar coordinates as

$$\mathbf{B}(r,\theta) = \frac{k}{r}\,\mathbf{e}_\theta \quad (r>0).$$

Hence the field is directed tangentially, and the tangential component of $\mathbf{B}$ on the circle of radius $r = a$ is k/a, which is constant. Hence the line integral around the circle is $(k/a)2\pi a = 2\pi k$.

(b) It is easiest to use the cylindrical form of curl given in Subsection 2.3. We can write $\mathbf{B}(\rho,\phi) = k\mathbf{e}_\phi/\rho$, so we have $B_\phi = k/\rho$ and $B_\rho = B_z = 0$. Thus, using equation (18), since the vector field is two-dimensional, we have

$$\mathbf{curl}\,\mathbf{B} = (-k/\rho^2 + k/\rho^2)\mathbf{e}_z = \mathbf{0}.$$

So part (a) shows that the vector field $\mathbf{B}$ is not a conservative field, since it has a non-zero line integral for the circle, but this part shows that $\mathbf{curl}\,\mathbf{B}$ is zero. This failure of the curl test occurs because the domain of $\mathbf{B}$ is not simply-connected. The origin, where $x^2 + y^2 = 0$, is excluded. Whenever there is a hole in the domain of a two-dimensional vector field, there is a possibility that the curl test will fail. You may take it that the curl test can be applied to all other vector fields that we ask you to work with in this module.

Unit 17

Multiple integrals

Introduction

You know that in one dimension the definite integral of a function $f(x)$ on an interval of the x-axis can be defined as the limit of a sum of function values on the interval. In Unit 16 you saw how this idea could be generalised to line integrals, and used to calculate properties of vector fields. In this unit we will generalise the idea of an integral still further to deal with two and three dimensions by introducing two new kinds of integrals, called *area integrals* and *volume integrals*.

Area integrals arise in calculations of quantities such as populations over regions. For example, suppose that you know the number of bacteria per unit area at each point on a glass plate. If this area density is uniform, then the total number of bacteria on any region of the plate is the product of the surface density and the area of that region. More generally, if the population density is non-uniform, then the total number of bacteria on a region of the plate can be expressed as an *area integral of the population density* over that region. Because a surface is a two-dimensional region, an area integral involves integrating with respect to two independent variables. Section 1 shows how area integrals can be evaluated as combinations of two ordinary integrals, while Section 2 describes applications of area integrals.

Volume integrals arise in the calculation of the mass of an object. For example, suppose that you know the mass per unit volume of a solid body. If the density is uniform, then the total mass of the body is the product of its density and its volume. More generally, the mass of the body is the *volume integral of the density* over the three-dimensional region occupied by the body. Section 3 shows how volume integrals can be expressed as combinations of three ordinary integrals.

Area integrals and volume integrals are examples of *multiple integrals*, the title of this unit. One of the major applications of multiple integrals is the calculation of *moments of inertia*. If you have not come across this term before, do not worry. Unit 21 will give a full treatment of moments of inertia and will describe their role in the rotational dynamics of solid bodies. In this unit we will give a definition of the moment of inertia but thereafter we will be concerned only with the techniques of calculating moments of inertia as multiple integrals.

Section 4 shows how area integrals can be used to compute the area of a curved surface.

1 Area integrals

In Unit 16 you saw that a scalar line integral involves integrating with respect to just one variable along a path. We now turn to *area integrals* in which the function being integrated is defined over an area, rather than along a path, so that it is a function of two variables.

This variable is usually the path parameter.

We can use an area integral to calculate, for example, the mass of a thin plate when the composition of the plate is non-uniform. Also, suppose that the height of a three-dimensional region Ω is represented by a height function $z = h(x, y)$ over an area S of the (x, y)-plane. Then the area integral of $h(x, y)$ over S will give the volume occupied by Ω.

We can also use area integrals to calculate populations. The distribution of, say, bacteria on a microscope slide can be described by a *population density* function giving the number of individuals per square metre at each point. Then the total population on the slide is the area integral of the population density over the slide. We will consider this and other applications of area integrals in Section 2. In this section we define area integrals (in Subsection 1.1) and see how to compute them using Cartesian coordinates (in Subsections 1.2 and 1.3).

1.1 Defining area integrals

Materials such as plate glass are usually sold by the square metre. By ignoring the fact that the material is a three-dimensional solid and treating it instead as an infinitely thin sheet, or **lamina**, we can describe the 'weight' of such a material by a *surface mass density* or mass per unit area of the sheet. The surface mass density, or **surface density** for short, is a constant for a uniform sheet, but will vary with position when the material is of non-uniform composition. If we imagine the lamina placed in the (x, y)-plane, then the surface density is a **surface density function** $f(x, y)$ defined on the region S of the (x, y)-plane occupied by the lamina.

We assume that we are carrying out a theoretical investigation rather than simply weighing the lamina to find its mass.

Suppose that we wish to find the mass of this lamina. We know that when $f(x, y)$ is a constant, the total mass of the lamina is simply the constant density (in $\mathrm{kg\,m^{-2}}$) times the total area (in $\mathrm{m^2}$) of the lamina. When the surface density varies, we can divide the region S into N small **area elements** such that the density can be considered constant on each element (see Figure 1). Let the area of the ith element be δA_i, and in this element select a point with coordinates (x_i, y_i). Then the mass of this element of the plate is approximately $f(x_i, y_i)\,\delta A_i$, and the total mass of the lamina is approximately the sum of the masses of all the elements:

$$\sum_{i=1}^{N} f(x_i, y_i)\,\delta A_i.$$

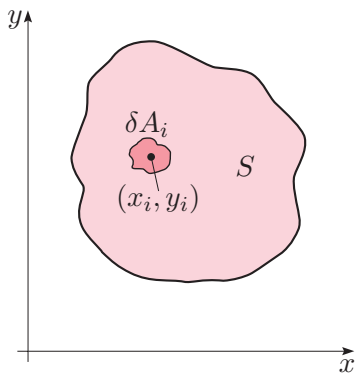


Figure 1 Lamina occupying region S of the (x, y)-plane

Now if we take the limit of this sum, as the number N of area elements increases indefinitely and the size of the elements tends to zero, we obtain an integral called an area integral, which we denote by

$$\int_S f(x, y)\,dA.$$

This area integral gives the mass of the lamina. Note that if we take $f(x, y) = 1$, then the area integral will give the area of the lamina.

We can define an area integral for any function of two variables over a region S in its domain.

Area integral

The **area integral** of $f(x, y)$ over a **region of integration** S in the (x, y)-plane, subdivided into N area elements where element i contains the point (x_i, y_i) and is of area δA_i, is

$$\int_S f(x, y)\, dA = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i)\, \delta A_i, \tag{1}$$

where $\delta A_i \to 0$ for each i as $N \to \infty$.

We will look at other applications of area integrals in Section 2. Next we consider how to evaluate area integrals, first over rectangular regions and then over non-rectangular regions. Since $f(x, y)$ is a function of two variables, we will find that the evaluation of an area integral involves evaluating *two* definite integrals, one after the other.

Area integrals are sometimes referred to as *double integrals* or *surface integrals*.

1.2 Area integrals over rectangular regions

In this subsection we consider how to apply equation (1) to evaluate an area integral, and in particular how to subdivide the region of integration S in a convenient way when working with Cartesian coordinates.

We begin by considering the case when the region S is the rectangle formed by the lines $x = a$, $x = b$ and $y = c$, $y = d$, as in Figure 2.

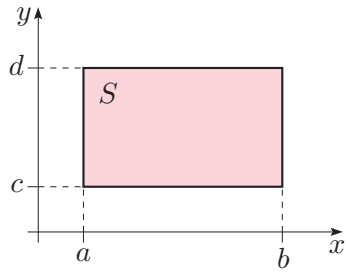


Figure 2 Rectangular region S

The most natural way to evaluate the sum in equation (1) is to divide the rectangle into small rectangular elements. We proceed as follows (see Figure 3).

- Divide the interval $[a, b]$ of the x-axis into n subintervals of widths $\delta x_1, \delta x_2, \ldots, \delta x_j, \ldots, \delta x_n$, and choose numbers $x_1, x_2, \ldots, x_n$ such that x_j is in the jth subinterval.
- Divide the interval $[c, d]$ of the y-axis into m subintervals of widths $\delta y_1, \delta y_2, \ldots, \delta y_k, \ldots, \delta y_m$, and choose numbers $y_1, y_2, \ldots, y_m$ such that y_k is in the kth subinterval.
- The area elements are the small rectangles so formed; the general element has side lengths $\delta x_j, \delta y_k$ and area $\delta x_j\, \delta y_k$, and contains the point (x_j, y_k).

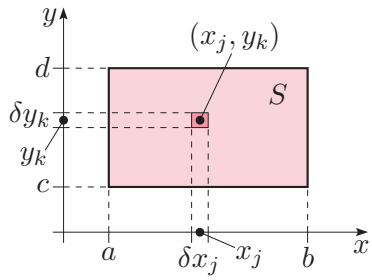


Figure 3 Small rectangular element within the region S

The rectangle S has been divided into $n \times m$ rectangular area elements, each one identified by a point (x_j, y_k), and we need to find a systematic way of including all $N = n \times m$ of them in the summation. Let us organise the rectangular area elements into thin strips. There are two straightforward ways. We can consider vertical strips of widths $\delta x_1, \delta x_2, \ldots, \delta x_j, \ldots, \delta x_n$ (as in Figure 4(a)), counting all the segments of length δy_k within each strip, or we can consider horizontal strips of widths $\delta y_1, \delta y_2, \ldots, \delta y_k, \ldots, \delta y_m$ (as in Figure 4(b)), counting all the segments of length δx_j in each.

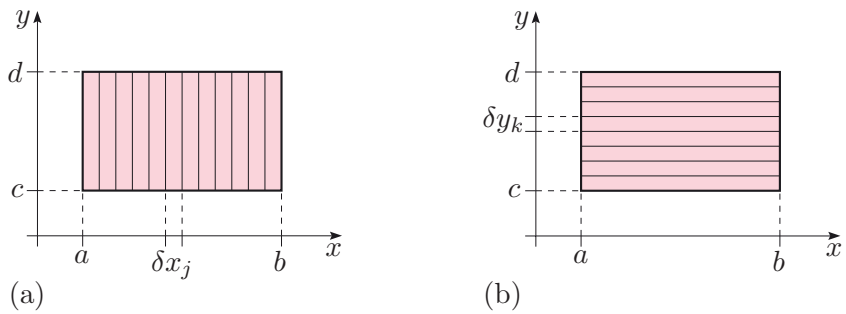


Figure 4 Rectangular region S divided into (a) vertical strips, (b) horizontal strips

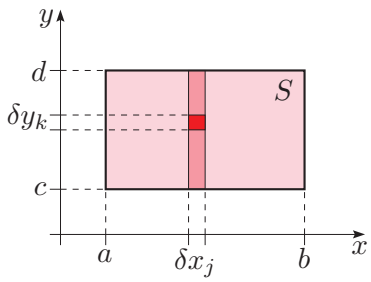


Figure 5 Rectangular region S showing the jth vertical strip

We choose vertical strips. The jth vertical strip (of width δx_j) is shown shaded in Figure 5. So a typical element with subscript i in equation (1) is included by virtue of its position y_k in the strip and the label x_j for the strip. Thus a typical term in the sum in equation (1) is $f(x_j, y_k)\,\delta x_j\,\delta y_k$. So the contribution to the sum from the m rectangular area elements in the jth vertical strip, of width δx_j, is

$$\left(\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right)\delta x_j.$$

In this summation, x_j and δx_j remain unchanged, and we are summing over the y_k terms, that is, over the subscript k. Now we need to add the contributions of the n vertical strips that lie between $x = a$ and $x = b$, so we are summing over the x_j terms, that is, over the subscript j. We then have an expression involving two summations:

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right)\delta x_j.$$

The summation over k is called the *inner summation*, and the summation over j is called the *outer summation*. The area integral of the function f over the region S is the limit of this double summation as the number of rectangles increases indefinitely, that is, as n and m increase indefinitely. So

$$\int_S f(x,y)\,dA = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\left(\lim_{m\to\infty}\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k\right)\delta x_j\right). \tag{2}$$

First let us look at the inner summation over k, remembering that x_j is constant here. In the limit as the lengths δy_k go to zero, this summation equals the definite integral of $f(x_j, y)$ between $y = c$ and $y = d$, that is, between the bottom and top of the jth vertical strip. We write

$$\lim_{m\to\infty}\sum_{k=1}^{m} f(x_j, y_k)\,\delta y_k = \int_{y=c}^{y=d} f(x_j, y)\,dy. \tag{3}$$

Remember that x_j is a constant in the summation and is held constant when we evaluate the integral. So the value of the integral will depend on x_j but not on y: that is, the value of the integral is a function of x_j. We illustrate this with an example.

You can think of the integration over y as *partial integration* by analogy with partial differentiation. You are integrating with respect to one variable while keeping the other variable fixed.

Example 1

Find the value of the definite integral

$$\int_{y=1}^{y=2} x_j y \, dy,$$

where x_j is a constant.

Solution

The function to be integrated is $f(x_j, y) = x_j y$. In the integration we are varying y (between the limits 1 and 2) but keeping x_j fixed. So

$$\int_{y=1}^{y=2} x_j y \, dy = x_j \int_{y=1}^{y=2} y \, dy = x_j \left[\tfrac{1}{2}y^2\right]_{y=1}^{y=2} = \tfrac{3}{2}x_j.$$

Suppose that we denote the value of the integral $\int_{y=c}^{y=d} f(x_j, y) \, dy$ in equation (3) by $g(x_j)$. Equation (2) then becomes

$$\int_S f(x, y) \, dA = \lim_{n\to\infty} \sum_{j=1}^{n} g(x_j) \, \delta x_j.$$

In this summation we are adding vertical strips between $x = a$ and $x = b$ (see Figure 6), and in the limit the summation approaches the definite integral of $g(x)$ between the limits $x = a$ and $x = b$:

$$\int_{x=a}^{x=b} g(x) \, dx = \lim_{n\to\infty} \sum_{j=1}^{n} g(x_j) \, \delta x_j.$$

In the limit of the double summation, as δx_j and δy_k go to zero in equation (2), the value of the area integral, in terms of two single integrals, the first over y and the second over x, is

$$\int_S f(x, y) \, dA = \int_{x=a}^{x=b} g(x) \, dx, \quad \text{where } g(x) = \int_{y=c}^{y=d} f(x, y) \, dy. \qquad (4)$$

The two integrals in equation (4) can be evaluated using standard techniques.

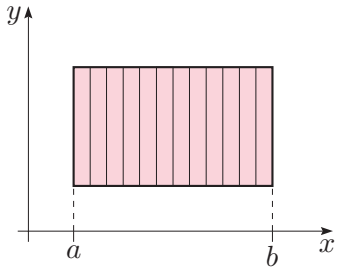


Figure 6 Rectangular region S divided into vertical strips

Area integral over a rectangular region

The area integral of a function $f(x, y)$ over a rectangular region S contained between the lines $x = a$, $x = b$ and $y = c$, $y = d$ is obtained as two successive integrals as follows:

$$\int_S f(x, y) \, dA = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} f(x, y) \, dy \right) dx.$$

Remember that in the integral over y, which is computed first, we treat x as a constant.

Note that the limits are stated with respect to the variable over which the integration is performed. This is done for clarity.

Example 2

Find the value of the area integral of the function $f(x, y) = xy$ over the rectangle bounded by the lines $x = 0$, $x = 3$ and $y = 1$, $y = 2$.

Solution

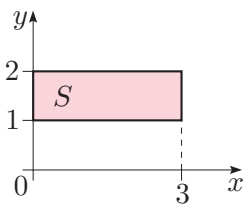


Figure 7 Region of integration

The region of integration S is shown in Figure 7. The area integral is

$$\int_S xy\,dA = \int_{x=0}^{x=3} \left(\int_{y=1}^{y=2} xy\,dy \right) dx.$$

The integral over y was evaluated in Example 1, that is,

$$\int_{y=1}^{y=2} xy\,dy = \tfrac{3}{2}x,$$

so

$$\int_S xy\,dA = \int_{x=0}^{x=3} \left(\int_{y=1}^{y=2} xy\,dy \right) dx = \int_{x=0}^{x=3} \tfrac{3}{2}x\,dx = \left[\tfrac{3}{4}x^2 \right]_0^3 = \tfrac{27}{4}.$$

The value of the area integral of xy over the rectangle is $\frac{27}{4}$.

Exercise 1

Find the value of the area integral of the function $f(x, y) = x^2y^3$ over the square bounded by the lines $x = 0$, $x = 2$ and $y = 1$, $y = 3$.

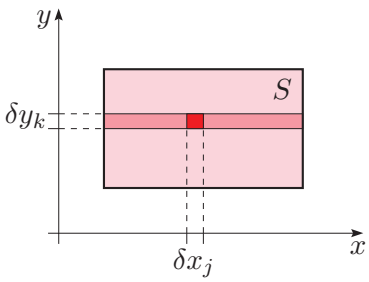


Figure 8 Rectangular region S showing the kth horizontal strip

So far we have evaluated area integrals by first integrating over y and then integrating over x. In the summations this is equivalent to summing first over k and then over j. However, we could have used another order for the summations, by first drawing a strip parallel to the x-axis of width δy_k containing y_k, and summing over the rectangles of width δx_j for all j running from 1 to n (see Figure 8). The result would be

$$\int_S f(x, y)\,dA = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} f(x, y)\,dx \right) dy.$$

This time in the inner integral we integrate over x, treating y as a constant, to give a function of y, and then we integrate over y, to complete the evaluation of the integral.

Exercise 2

Find the value of the area integral in Example 2, but integrate over x first and then over y.

The value of the area integral in Exercise 2 is the same as in Example 2, which you may have expected since the function and the region of integration are the same. Either order of evaluating the single integrals will give the same answer.

1.3 Area integrals over non-rectangular regions

The area integral of a function over a rectangular region involves two single integrals for which the limits of integration are constants. For non-rectangular regions, we must be more careful in setting up the integrals because the strips are no longer of the same length and the limits on the inner integral depend on the variable in the outer integral. To illustrate this, consider the following example.

Example 3

Find the value of the area integral of the function $f(x, y) = xy$ over the region S bounded by the curves $y = 2x$, $y = x^2$ and the line $x = 1$.

Solution

We begin by drawing a diagram to show the region of integration – it is the shaded region in Figure 9. We choose to integrate first over y and then over x since the geometry of the region of integration suggests that this is likely to lead to the easier calculation. If we choose to integrate over x first, then the nature of the upper limit of the x-integral would depend on whether $y < 1$ or $y > 1$. We would therefore need to break up the region of integration into two subregions, one with $0 \leq y \leq 1$ and another with $1 \leq y \leq 2$. So, to avoid this, we integrate over y first.

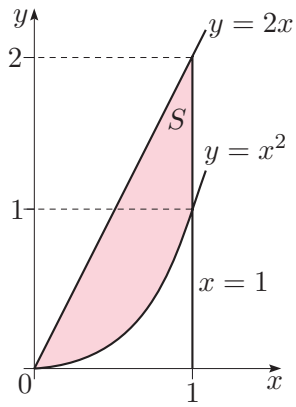


Figure 9 Region of integration

To decide on the limits of the integration over y, consider a vertical strip drawn at an arbitrary value of x, as shown in Figure 10(a). The ends of the strip lie on the curves $y = x^2$ and $y = 2x$. Hence the lower and upper limits for the y-integration are (the functions) $y = x^2$ and $y = 2x$, respectively. The y-integration sums all area elements in the strip of Figure 10(b).

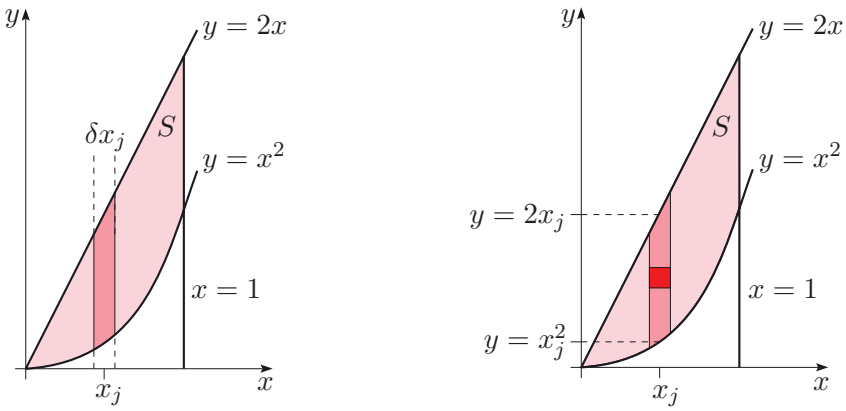


Figure 10 (a) Vertical strip at $x = x_j$ in the region of integration. (b) An area element in the strip.

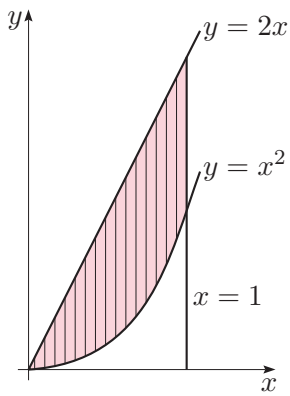


Figure 11 Vertical strips dividing the region of integration

The x-integration sums over all the strips (see Figure 11). The first strip is at $x = 0$, and the last strip is at $x = 1$. Hence the lower and upper limits for the x-integration are $x = 0$ and $x = 1$, respectively, and in the limit equation (2) becomes

$$\int_S xy\,dA = \int_{x=0}^{x=1} \left(\int_{y=x^2}^{y=2x} xy\,dy \right) dx. \tag{5}$$

The inner integral, in the brackets, is an integral with respect to y and can be evaluated by treating x as a constant, to give

$$\int_{y=x^2}^{y=2x} xy\,dy = x\left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=2x} = x\left(\tfrac{1}{2}(2x)^2 - \tfrac{1}{2}(x^2)^2\right) = 2x^3 - \tfrac{1}{2}x^5.$$

Notice that the result of evaluating this integral is a function of x, so the area integral in equation (5) is reduced to a single integral over x, and

$$\int_S xy\,dA = \int_{x=0}^{x=1} \left(2x^3 - \tfrac{1}{2}x^5\right) dx = \left[\tfrac{2}{4}x^4 - \tfrac{1}{12}x^6\right]_0^1 = \tfrac{5}{12}.$$

In Example 3, the limits on the integral over y depend on the variable x. Drawing a diagram of the region of integration is very helpful for getting the limits correct, and the method of solution used in the example works for all area integrals. We will now generalise the procedure for evaluating area integrals.

Consider a non-rectangular region S like one of the two shown in Figure 12.

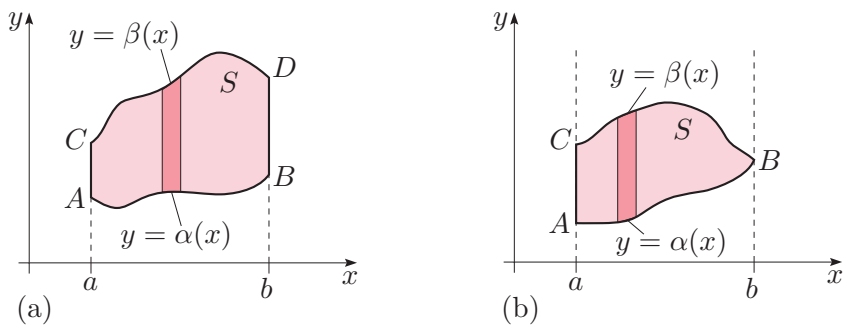


Figure 12 Non-rectangular regions of integration S. In (a) the maximum value of x on the boundary of S lies along a line, whereas in (b) it is at a point.

The minimum and maximum values can be straight lines, as shown in Figure 12(a), or points. One such point is shown in Figure 12(b).

Suppose that a and b are the minimum and maximum values of x for the points on the boundary of S. Let $y = \alpha(x)$ and $y = \beta(x)$ be the equations of the boundary curves AB and CD, respectively, as shown in Figure 12(a). (In Figure 12(b), points B and D coincide, so in this case, $y = \beta(x)$ is the equation of the boundary curve CB.) Then

$$\int_S f(x,y)\,dA = \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} f(x,y)\,dy \right) dx.$$

The area integral can be written as two single definite integrals, but we must be careful with the limits. The limits on the inner integral are functions of x rather than constants.

We can summarise the steps for evaluating an area integral using vertical strips, as follows.

Procedure 1 Evaluating an area integral (integration with respect to y first)

To evaluate an area integral

$$\int_S f(x, y)\, dA,$$

carry out the following steps.

1. Draw a diagram showing the region of integration S.
2. Draw on the diagram a strip parallel to the y-axis, and show the lower limit $y = \alpha(x)$ and the upper limit $y = \beta(x)$ of this strip. These are the limits of the y-integration, that is, for the 'inner' integral.
3. Determine the lower limit a and upper limit b of x for points on the boundary of S. These are the limits of the x-integration, that is, for the 'outer' integral.
4. Write the area integral as two single integrals, making sure that the outer limits are constants:
$$\int_S f(x, y)\, dA = \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\, dy \right) dx.$$
5. Evaluate the inner integral, holding x constant, to give
$$g(x) = \int_{y=\alpha(x)}^{y=\beta(x)} f(x, y)\, dy.$$
6. Evaluate the outer integral to give
$$\int_S f(x, y)\, dA = \int_{x=a}^{x=b} g(x)\, dx.$$

Note that there should be no holes in region S, and that $\alpha(x)$ and $\beta(x)$ should be single-valued functions.

Exercise 3

Use Procedure 1 to find the value of the area integral of the function $f(x, y) = y$ over the region S bounded by the curves $y = x^2$ and $y = x + 2$.

Exercise 4

Find the value of the area integral of the function $f(x, y) = x - y$ over the triangle bounded by the lines $y = x - 1$, $x = 3$ and $y = 0$.

Exercise 5

Find the value of the area integral of the function $f(x, y) = x + y$ over the triangle S bounded by the lines $y = 1 - x$, $x = 0$ and $y = 0$.

In the course of this unit, you will often come across single integrals that contain both a function $f(x)$ and its derivative $f'(x)$, and can be cast in the form

$$\int k\, f'(x) \left(f(x)\right)^n dx.$$

There is a trick to integrate this, based on recognising that, from the chain rule,

$$\frac{d}{dx}\left(f(x)\right)^{n+1} = (n+1)\left(f(x)\right)^n f'(x),$$

so

$$\int k\, f'(x) \left(f(x)\right)^n dx = \frac{k}{n+1}\left(f(x)\right)^{n+1}. \tag{6}$$

Exercise 6

Find the value of the area integral of the function $f(x, y) = x$ over the quarter-disc S given by $x^2 + y^2 \leq 1$ $(x \geq 0,\ y \geq 0)$.

Exercise 7

Determine the region of integration for, and evaluate,

$$\int_{x=1}^{x=2} \left(\int_{y=x^2}^{y=x+2} 1\, dy \right) dx.$$

Procedure 1 is easily adapted for this reversed order of integration by interchanging x and y.

So far we have chosen to organise the area elements into vertical strips and integrate over y first and then x. We can reverse the order of integration, carrying out the x-integration first and then the y-integration by imagining the area divided into horizontal strips. The following example illustrates how this is done.

Example 4

Find the value of the area integral of the function xy^2 over the region S bounded by the curve $y = x^2$ and the line $y = x$ by integrating first over x and then over y.

Solution

The region of integration is shown in Figure 13(a), and one horizontal strip is shown in Figure 13(b).

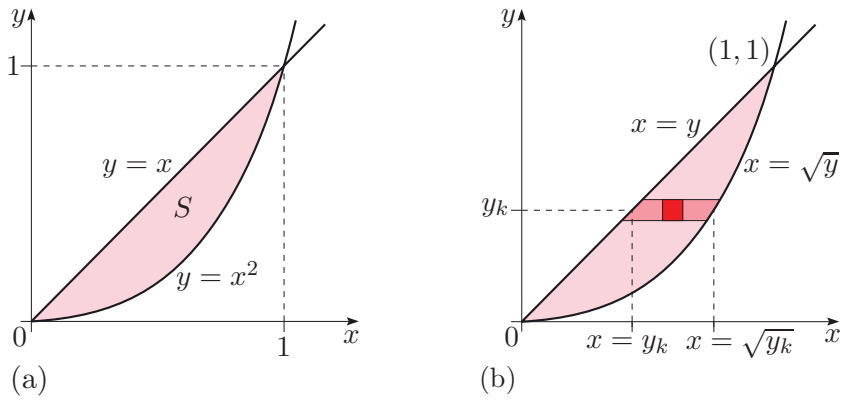


Figure 13 (a) Region of integration S. (b) The kth horizontal strip.

The x-integral effectively sums area elements along each horizontal strip. To decide on the limits for the x-integral, look at the left-hand and right-hand endpoints of the strip. These endpoints lie on the graphs of $x = y$ and $x = \sqrt{y}$, so the lower and upper limits of the x-integral are the functions $x = y$ and $x = \sqrt{y}$, respectively. The limits for the y-integral are the y-values where the graphs intersect, where $y = x = x^2$, that is, where $x^2 - x = x(x-1) = 0$. These points of intersection are therefore $(0, 0)$ and $(1, 1)$, as can be seen in Figure 13(a). Hence the area integral is

Note that we take the *positive* root of $y = x^2$ here as we have $x > 0$.

$$\int_S f(x,y)\,dA = \int_{y=0}^{y=1} \left(\int_{x=y}^{x=\sqrt{y}} xy^2\,dx \right) dy.$$

The x-integral can be evaluated, treating y as a constant, to give

$$\int_{x=y}^{x=\sqrt{y}} xy^2\,dx = y^2 \left[\tfrac{1}{2}x^2\right]_{x=y}^{x=\sqrt{y}} = \tfrac{1}{2}(y^3 - y^4),$$

so evaluation of the area integral is reduced to a single integral in y, giving

$$\int_S f(x,y)\,dA = \int_{y=0}^{y=1} \tfrac{1}{2}(y^3 - y^4)\,dy = \left[\tfrac{1}{8}y^4 - \tfrac{1}{10}y^5\right]_{y=0}^{y=1} = \tfrac{1}{40}.$$

Exercise 8

Evaluate

$$\int_S (x^2 + y^2)\,dA,$$

where S is the triangle formed by the lines $y = 0$, $y = x - 1$ and $x = 2$, in the following ways.

(a) Integrate over y first, then over x.

(b) Integrate over x first, then over y.

Exercise 8 illustrates that either order of integration is acceptable, although the calculations are often easier for one particular order. It is important to be careful with the limits on the two single integrals. The limits on the outer integral are constants, whereas the limits on the inner integral will, in general, be non-constant functions. Only for rectangular regions of integration will all four limits be constants.

Exercise 9

By performing the integration with respect to x first, integrate $f(x, y) = x$ over the region bounded by the positive x-axis and the parabolas $y = x^2$ and $y = 4 - x^2$.

2 Applications of area integrals

Surface density functions were mentioned at the beginning of Section 1.

In Section 1 we were concerned with the mathematical problem of evaluating area integrals. In this section we consider applications of area integrals to a variety of problems involving surface density functions. Some of the applications involve integrating over a circular region of the (x, y)-plane, and in such cases it is usually easier to evaluate these area integrals using polar coordinates, as we demonstrate in Subsection 2.2. In Subsection 2.3, we introduce the idea of the *moment of inertia* of a body, a subject of great importance for Unit 21, and show how the moments of inertia of laminas can be calculated as area integrals.

2.1 Integrating surface density functions

Surface density functions can describe surface distributions of mass, populations and many other scalar quantities. The area integral of a surface density function then gives the total mass, population or whatever of the surface.

Example 5

A population of bacteria is grown on a rectangular glass plate. The plate is placed in the (x, y)-plane with its edges on the lines $y = 0$, $y = 0.04$ and $x = 0$, $x = 0.03$ (see Figure 14).

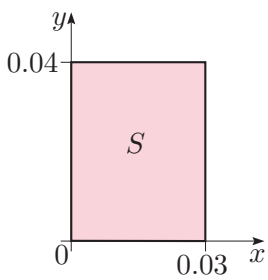


Figure 14 Rectangular glass plate

Determine the total number N_b of bacteria on the plate, given that the distribution of the population (in bacteria per square metre) is described by the surface density function

$$f(x, y) = (1 - 1000x^2) \times 10^{11}.$$

Solution

We integrate the surface density function over the rectangular region S occupied by the plate, choosing arbitrarily to integrate over x first. Thus

$$
\begin{aligned}
N_{\mathrm{b}} &= \int_S f(x,y)\,dA \\
&= 10^{11} \int_{y=0}^{y=0.04} \left(\int_{x=0}^{x=0.03} (1 - 1000x^2)\,dx \right) dy \\
&= 10^{11} \int_{y=0}^{y=0.04} \left[x - \tfrac{1000}{3}x^3 \right]_{x=0}^{x=0.03} dy \\
&= 10^{11} \int_0^{0.04} 0.021\,dy = 8.4 \times 10^7.
\end{aligned}
$$

Exercise 10

The density of bacteria on the surface of a glass plate in the (x, y)-plane is given by

$$f(x,y) = A\left(2 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right) \quad (x^2 + y^2 < 2a^2),$$

where A and a are positive constants.

How many bacteria are there on a triangular region of the plate bounded by the x- and y-axes and the line $y = a - x$, when $A = 10^{12}$ and $a = 0.01$?

(In this exercise the constants are represented by symbols that can be used throughout the calculation. You are advised to do this, putting in the numbers only at the end in order to answer the question. The symbolic answer gives a more general result, which could be used with other values of the constants.)

Suppose for the bacteria in Exercise 10 that we want to know the number within a circular region S of radius a centred on the origin, as shown in Figure 15. Let us choose to integrate with respect to y first. The lower and upper ends of a vertical strip lie on the lower and upper semicircles

$$y = -\sqrt{a^2 - x^2} \quad \text{and} \quad y = \sqrt{a^2 - x^2}.$$

These functions are the limits of the y-integration. The x-integral represents the addition of vertical strips starting at $x = -a$ and ending at $x = a$. Hence the total number of bacteria within the circle (over the disc) is

$$\int_S f\,dA = A \int_{x=-a}^{x=a} \left(\int_{y=-\sqrt{a^2-x^2}}^{y=\sqrt{a^2-x^2}} \left(2 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right) dy \right) dx. \qquad (7)$$

Evaluating the integral in equation (7) is not particularly difficult, but it is more difficult than it needs to be. You will see in the next subsection that this double integral, and many other area integrals over circular regions, become much easier when we use polar coordinates. As you will see in Example 6, the total number of bacteria in this case is given by $N_{\mathrm{b}} = \frac{3}{2}\pi A a^2$.

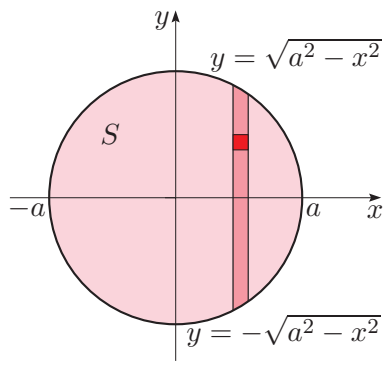


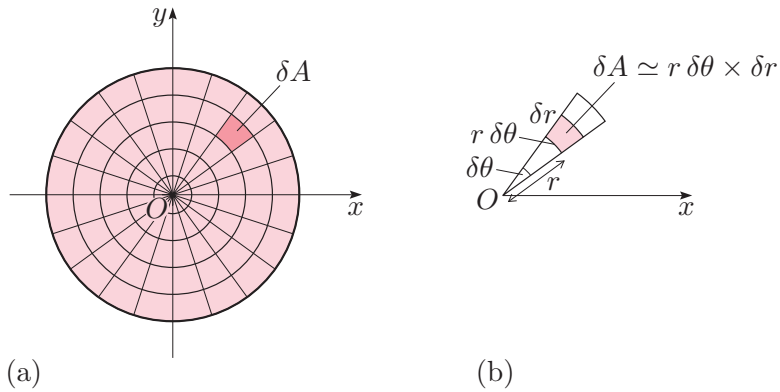
Figure 15 Circular region of integration, showing a vertical strip

2.2 Changing to polar coordinates

In order to see how to set up an area integral in polar coordinates, recall first how a rectangular region was divided up in Cartesian coordinates by a rectangular grid of lines, $x =$ constant and $y =$ constant. A small rectangular element, the **Cartesian area element**, has area $\delta A = \delta y\, \delta x$. This element is represented symbolically as $dy\, dx$ in the area integral

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx.$$

Now consider the case of polar coordinates r and θ. Figure 16(a) shows the subdivision of a circular region (disc) by a grid of circles $r =$ constant and 'spokes' $\theta =$ constant.



(a) (b)

Figure 16 (a) Circular region divided into area elements using a grid based on polar coordinates. (b) A single area element shown in more detail.

As illustrated in Figure 16(b), each *area element* is nearly a rectangle of area $\delta A \simeq r\, \delta\theta \times \delta r$, and this approximation improves as δr and $\delta\theta$ tend to zero. To appreciate this, recall that an arc length is given by the radius multiplied by the subtended angle, so here the arc length is $r\, \delta\theta$, hence the area is $\delta A \simeq r\, \delta\theta\, \delta r$. The value of a function f at a point of this element is $f(r, \theta)$, so the contribution of this area element to the area integral is approximately $f(r, \theta)\, r\, \delta r\, \delta\theta$. Summing over all area elements and taking the limit in equation (1), we have the following definition.

Area integral in polar coordinates

The area integral of a function $f(r, \theta)$ over a disc D of radius a is

$$\int_D f\, dA = \int_{\theta=-\pi}^{\theta=\pi} \left(\int_{r=0}^{r=a} f(r, \theta)\, r\, dr \right) d\theta. \qquad (8)$$

The factor r comes from the approximation $r\, \delta r\, \delta\theta$ of the area of the area element. It is very easy to forget this factor!

We can choose to integrate with respect to r first and then θ as in equation (8), or we can reverse the order of integration, as is convenient.

Because the limits are constants, they will not change when the order of integration is reversed.

Example 6

Evaluate the number of bacteria on the circular region of the (x, y)-plane of radius a centred at the origin when the surface density of bacteria is given by

$$f(x, y) = A\left(2 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right).$$

What is the number N_b of bacteria in the circle when $A = 10^{12}$ and $a = 0.01$?

Solution

We first express the surface density function in polar coordinates. Recognising that $x^2 + y^2 = r^2$, we have $f(r, \theta) = A(2 - r^2/a^2)$. The region of integration is a disc of radius a centred on the origin. Using equation (8), we can express the number of bacteria as

Note the deliberate abuse of notation where $f(x, y)$, in Cartesian coordinates, is written $f(r, \theta)$ in polar coordinates.

$$N_\mathrm{b} = A\int_{\theta=-\pi}^{\theta=\pi}\left(\int_{r=0}^{r=a}\left(2 - \frac{r^2}{a^2}\right) r\,dr\right) d\theta. \tag{9}$$

The inner integral, the integral with respect to r with θ constant, represents summing elements of length δr along the narrow sector shown in Figure 16(b). The θ-integral then represents the summing of all sectors of the circle.

The r-integral gives

$$\int_{r=0}^{r=a}\left(2 - \frac{r^2}{a^2}\right) r\,dr = \left[r^2 - \frac{r^4}{4a^2}\right]_0^a = \tfrac{3}{4}a^2.$$

The area integral in equation (9) is now reduced to a single integral, so

$$N_\mathrm{b} = \tfrac{3}{4}Aa^2\int_{-\pi}^{\pi} d\theta = \tfrac{3}{4}Aa^2\big[\theta\big]_{-\pi}^{\pi} = \tfrac{3}{2}\pi Aa^2.$$

For the given values of A and a, the total number N_b of bacteria in the circular region is 4.7×10^8 to two significant figures.

There are two points to notice about Example 6. First, the area integral has constant limits and is very easy to evaluate compared with the Cartesian equivalent, equation (7). Second, the function $f(r, \theta)$ has circular symmetry. It is independent of the angular coordinate θ and depends on r only, and consequently the integration over θ in the area integral yields the factor

$$\int_{-\pi}^{\pi} d\theta = \big[\theta\big]_{-\pi}^{\pi} = 2\pi.$$

Thus for any function $f(r, \theta)$ that varies with r only, the area integral over a disc D of radius a centred on the origin is

$$\int_D f \, dA = 2\pi \int_0^a f(r) \, r \, dr, \tag{10}$$

in which we have written $f(r)$ for $f(r, \theta)$ since f depends on r only.

Exercise 11

A thin circular magnifying glass of radius a can be modelled as a thin disc with a surface mass density given by $f(r, \theta) = k(1 - r^2/a^2)$, where the constant k is the surface density at the centre.

Determine the total mass of the lens when $a = 2.5 \times 10^{-2}$ and $k = 3$.

Exercise 12

Consider the integral

$$J = \int_{-\infty}^{\infty} \exp(-x^2) \, dx,$$

called the *Gaussian integral*, which plays an important role in probability theory and statistics.

(a) Show that J^2 is the area integral of $\exp\left(-(x^2 + y^2)\right)$ over the whole of the (x, y)-plane.

(b) Use polar coordinates to evaluate J^2, and hence show that $J = \sqrt{\pi}$.

Exercise 13

A drop of coloured chemical falls onto a plane sheet of blotting paper and spreads to form a stain. Taking the origin to be the point where the drop hits the blotting paper, the density of the chemical in the stain is described by $f(r, \theta) = A \exp(-r^2/a^2)$, in polar coordinates, where $A = 10^{-5}$ (in $\mathrm{kg\,m^{-2}}$) and $a = 1.5 \times 10^{-2}$ (in m).

Find the mass of chemical within a radius of $b = 0.01$ from the origin.

2.3 Moments of inertia

We now turn to an important application of area integrals, the calculation of *moments of inertia* for laminas. The central role played by moments of inertia in the dynamics of rotating bodies is fully described in Unit 21. Here, and in the next section, we are concerned only with calculations of moments of inertia using multiple integrals, so we give only a minimal introduction to the topic.

Figure 17 shows a particle of mass m moving in a plane with constant speed v along a circular path of radius d centred on a point O. If O is the origin of the (x, y)-plane, then such a motion can be thought of as rotation about the z-axis. Circular motion will be discussed more fully in Unit 20, but for the purpose here, you should note the following. If θ is the angle (in radians) subtended by an arc of a circle of radius d, then θd is the length of the arc. Therefore $v = |\dot{\theta}|d$, since d is constant for circular motion, so $v = \omega d$, where $\omega = |\dot{\theta}|$, the *angular speed* of the particle.

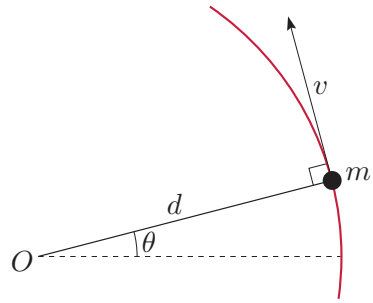


Figure 17 Particle moving in a circular path

You know that the kinetic energy of the particle is $\frac{1}{2}mv^2$. Now we can write this as $\frac{1}{2}m\omega^2 d^2$, where $\omega = v/d$ is the angular speed.

We make the following definition.

Moment of inertia of a particle

The quantity $I = md^2$, the product of a particle's mass m and the square of its perpendicular distance d from a fixed axis, is called the **moment of inertia** of the particle about the axis.

So the kinetic energy of the particle can also be written as $\frac{1}{2}mv^2 = \frac{1}{2}I\omega^2$. Comparing this with the more familiar $\frac{1}{2}mv^2$, you can see that if the motion of the particle is described by an angular speed ω rather than a speed v, then I replaces m as the parameter describing the inertia of the particle.

Consider now an N-particle system rotating about an axis. Each particle rotates in a circle of radius given by its perpendicular distance from the axis (see Figure 18). The ith particle of the system has moment of inertia $m_i d_i^2$, where m_i is the mass of the particle and d_i is its perpendicular distance from the axis. So the moment of inertia of the N-particle system about the axis is

$$I = \sum_{i=1}^{N} m_i d_i^2. \tag{11}$$

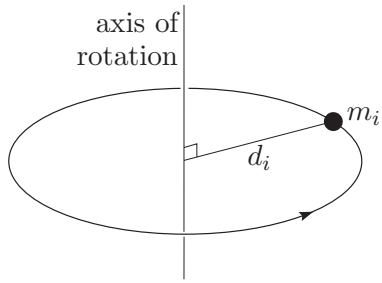


Figure 18 Circular motion of the ith particle

In Unit 21 we will be interested in the motion of rigid bodies that can rotate about fixed axes, and we will need to consider, for each rigid body, its moment of inertia. We can define the *moment of inertia of a rigid body* about an axis to be the limit of the sum of the moments of inertia of its N constituent particles when $N \to \infty$. Thus, using equation (11), the moment of inertia of the rigid body about the fixed axis is

$$I = \lim_{N\to\infty} \sum_{i=1}^{N} m_i d_i^2. \tag{12}$$

If the rigid body is a lamina in the (x, y)-plane that can rotate about the z-axis, then any variation of composition can be described by a surface density function $f(x, y)$.

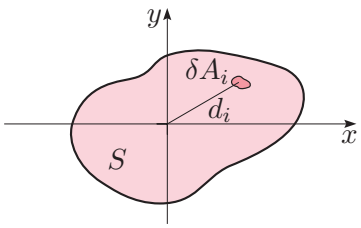


Figure 19 Lamina in the (x, y)-plane

We subdivide the lamina into a large number N of small area elements δA_i, modelled as particles (see Figure 19). Each element has mass $\delta m_i = f_i\,\delta A_i$, where f_i is the surface density at a point in the area element δA_i. The moment of inertia of this element about the z-axis is $\delta m_i\,d_i^2 = f_i\,\delta A_i\,d_i^2$, where d_i is the distance of the element from the origin. From equation (12), the moment of inertia of the whole lamina about the z-axis is

$$I = \lim_{N\to\infty} \sum_{i=1}^{N} f_i\,\delta A_i\,d_i^2,$$

the area integral of fd^2, where $d(x, y)$ is a function giving the distance of the point (x, y) from the axis of rotation. Thus if a lamina occupies a region S of the (x, y)-plane and has surface density function f, then its moment of inertia about the z-axis is given as follows.

Moment of inertia of a lamina

The moment of inertia about the z-axis of a lamina with surface density function f occupying a region S of the (x, y)-plane is

$$I = \int_S fd^2\,dA, \tag{13}$$

where d is the perpendicular distance from the z-axis, so

$$fd^2 = \begin{cases} f(x,y)\,(x^2+y^2) & \text{in Cartesian coordinates,} \\ f(r,\theta)\,r^2 & \text{in polar coordinates.} \end{cases} \tag{14}$$

Example 7

Determine the moment of inertia of a square plate of side length a, constant surface density f, and negligible thickness, about an axis that is perpendicular to the plane of the plate, passing through its centre.

Solution

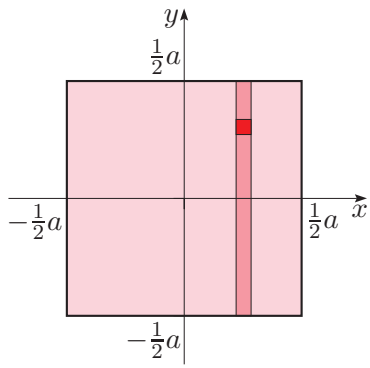


Figure 20 Square plate in the (x, y)-plane

Figure 20 shows the plate or lamina in the (x, y)-plane with its centre at the origin. Since the region of integration is a square, we use Cartesian coordinates. The surface density function f is constant, so we can take it outside the area integral of equation (13), and we have

$$\begin{aligned} I &= f\int_{x=-a/2}^{x=a/2}\left(\int_{y=-a/2}^{y=a/2}(x^2+y^2)\,dy\right)dx \\ &= f\int_{x=-a/2}^{x=a/2}\left[x^2y+\tfrac{1}{3}y^3\right]_{y=-a/2}^{y=a/2}dx \\ &= 2f\int_{-a/2}^{a/2}\left(\tfrac{1}{2}x^2a+\tfrac{1}{24}a^3\right)dx \\ &= 2f\left[\tfrac{1}{6}ax^3+\tfrac{1}{24}a^3x\right]_{-a/2}^{a/2} = \tfrac{1}{6}fa^4. \end{aligned}$$

The mass of the plate is $M = fa^2$, so we can express the moment of inertia of the plate as $I = \frac{1}{6}Ma^2$.

Exercise 14

Determine the moment of inertia of a flat ruler of length $a = 0.30$ and width $b = 0.04$ about an axis that is perpendicular to the plane of the ruler, passing through its centre. Assume that the mass of the ruler can be modelled by a constant surface density function $f = 10^{-2}$.

The ruler is considered as a lamina in that its thickness is assumed to be negligible.

(*Hint*: Evaluate the area integral using the symbols a, b and f, then substitute the numbers into the answer.)

It is usual to quote the moment of inertia of a body in terms of its total mass M and its dimensions. For a square of side length a, the moment of inertia about an axis through its centre that is perpendicular to the plane of the square is

We did this in Example 7 and in the solution to Exercise 14. Note that the SI units of moment of inertia are $\mathrm{kg\,m^2}$.

$$I_{\text{square}} = \tfrac{1}{6}Ma^2, \tag{15}$$

and for a rectangle of length a and width b, the moment of inertia about an axis through its centre that is perpendicular to the plane of the rectangle is

$$I_{\text{rectangle}} = \tfrac{1}{12}M(a^2 + b^2). \tag{16}$$

This last result can be used to find the moment of inertia of a thin rod of mass M and length a about an axis through its centre at right angles to its length. We take the limit as $b \to 0$ in equation (16) to obtain

$$I_{\text{rod}} = \tfrac{1}{12}Ma^2. \tag{17}$$

We used Cartesian coordinates in Example 7 and Exercise 14 because the regions of integration were rectangular. To find the moment of inertia of a disc, it is simpler to use polar coordinates.

Example 8

Express the moment of inertia of a disc of radius a with surface density function f, about an axis perpendicular to its plane and passing through its centre, as an area integral in polar coordinates.

Solution

Using polar coordinates in equation (13), which entails replacing f in equation (8) by fr^2, the moment of inertia of the disc about its centre is given by the area integral

$$I_{\text{disc}} = \int fr^2\,dA = \int_{\theta=-\pi}^{\theta=\pi}\left(\int_{r=0}^{r=a} fr^3\,dr\right)d\theta.$$

Here we have left the surface density function f inside the integral in case it is not a constant.

Exercise 15

Evaluate the area integral in Example 8 for the case of a uniform disc of radius a, and hence show that the moment of inertia of the disc about an axis perpendicular to its plane and passing through its centre is $\frac{1}{2}Ma^2$, where M is the mass of the disc.

Exercise 16

Find the moment of inertia of a flat circular washer of uniform material, negligible thickness and total mass M, in the shape of an annulus of internal radius a and external radius b, about an axis perpendicular to the plane of the washer through its centre.

Exercise 17

Determine the moment of inertia about the z-axis of the magnifying glass in Exercise 11, which has surface density

$$f(r, \theta) = k(1 - r^2/a^2),$$

where k is a constant, and a is the radius of the magnifying glass.

Laminas, by definition, have negligible thickness.

In the above examples and exercises we have modelled plates and discs as laminas with variations of composition described by surface density functions. In an improved model of a plate, we recognise that the thickness cannot be ignored and model the plate as a solid body, forming a *volume integral* by summing over small *volume elements* of the body. We do this in Section 3.

3 Volume integrals

In Section 2 you saw how to calculate the masses and moments of inertia of laminas where the distribution of mass was modelled by a surface density function defined on a region of the (x, y)-plane. A more fundamental approach is to recognise at the outset that a real body is a three-dimensional object. Masses and moments of inertia are then expressed as *volume integrals*. In this section we define the volume integral and show how volume integrals are evaluated and used to find masses and moments of inertia.

3.1 Defining and evaluating volume integrals

You know that the mass of an object of uniform density is simply the product of its volume times its density, where the *density* here is the *mass per unit volume*, measured in $\mathrm{kg\,m^{-3}}$. To find the mass of an object of non-uniform density, we subdivide the object into very small **volume elements**, estimate the mass of an element as the product of its volume times the local value of the density, form the sum over all elements, and then go to the limit of an infinitely large number of small elements. The summation then becomes an integral called the *volume integral*, which we can evaluate.

Consider a scalar field $f(x, y, z)$ defined in a three-dimensional region B, and subdivide B into N elements, where element i has volume δV_i. We make the following definition.

Volume integral

The **volume integral** of $f(x, y, z)$ over a **region of integration** B, subdivided into N elements where element i contains the point (x_i, y_i, z_i) and is of volume δV_i, is

$$\int_B f(x, y, z)\, dV = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i, y_i, z_i)\, \delta V_i, \tag{18}$$

where $\delta V_i \to 0$ as $N \to \infty$.

If $f(x, y, z) = 1$, then we obtain the volume of region B.

Before we can evaluate a volume integral, we must choose a shape for the volume elements. The shape that we choose depends on whether it is best to work in the Cartesian, cylindrical or spherical coordinate system. We begin with the Cartesian system.

Volume integral over a rectangular cuboid

The volume integral of a function $f(x, y, z)$ over a rectangular cuboid B whose faces lie in coordinate planes $x = a$, $x = b$, $y = c$, $y = d$, $z = p$, $z = q$ is obtained as three successive integrals as follows:

$$\int_B f(x, y, z)\, dV = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \left(\int_{z=p}^{z=q} f(x, y, z)\, dz \right) dy \right) dx. \tag{19}$$

A *rectangular cuboid* is just a rectangular block.

The method for evaluating the volume integral is similar to that for evaluating surface integrals over rectangular regions in the (x, y)-plane, except that now we must integrate over three variables. The method is best illustrated by an example.

Volume integrals are sometimes referred to as *triple integrals*.

Example 9

The density inside a cube whose faces are the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ is given by the density function

$$f(x, y, z) = c(x^2 + y^2 + z^2),$$

where c is a constant.

Determine the mass of the cube.

Solution

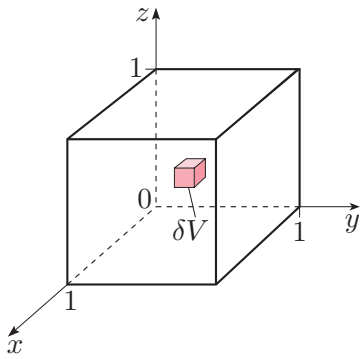


Figure 21 Cubic region of integration

The mass of the cube is given by the volume integral

$$\int_B f\,dV = \int_B c(x^2 + y^2 + z^2)\,dV.$$

The region of integration B is shown in Figure 21. To evaluate this volume integral, we form three single integrals. The method for finding the limits for these single integrals is similar to the approach that we used for area integrals. The volume element in Cartesian coordinates is a small block with volume $\delta V = \delta x\,\delta y\,\delta z$. If we draw within the region a vertical column of rectangular cross-section, then the volume of the column is found by summing the volume elements along this column. This column intersects the bottom and top faces of the region at the lower and upper limits of the z-integration. In this case, the limits are $z = 0$ and $z = 1$ (see Figure 22(a)).

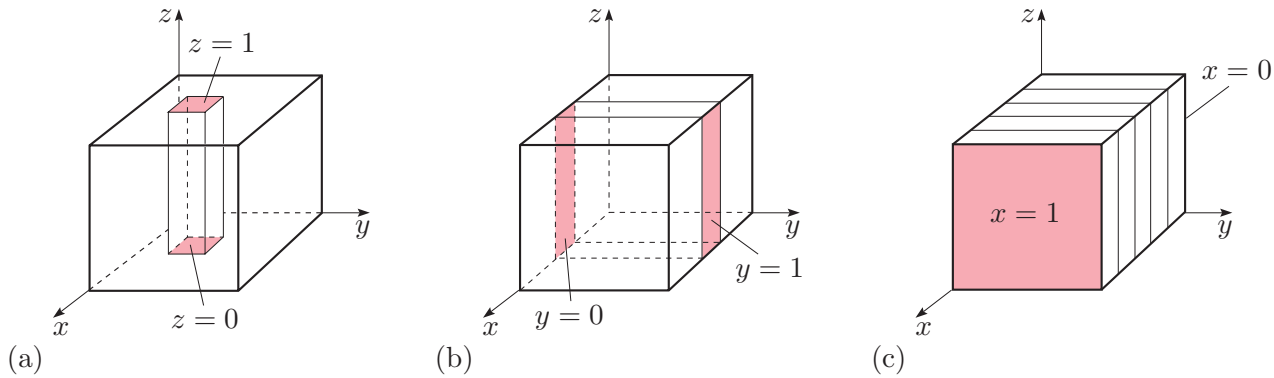


Figure 22 Cubic region of integration showing (a) a vertical column, (b) a slice parallel to the (y, z)-plane, and (c) the region divided into slices parallel to the (y, z)-plane

Now we form the y-integral by summing all the columns in a 'slice' parallel to the (y, z)-plane. The limits of the y-integration are therefore $y = 0$ and $y = 1$ (see Figure 22(b)). Finally, we complete the volume by adding together all possible slices so that the limits of the x-integration are $x = 0$ and $x = 1$ (see Figure 22(c)).

Hence the mass M of the cube can be found from three successive integrals:

$$M = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} \left(\int_{z=0}^{z=1} c(x^2 + y^2 + z^2)\,dz \right) dy \right) dx.$$

The inner integral is evaluated first. This is an integral over z with x and y treated as constants. Hence

$$M = \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} c\left[(x^2+y^2)z + \tfrac{1}{3}z^3\right]_{z=0}^{z=1} dy \right) dx$$
$$= \int_{x=0}^{x=1} \left(\int_{y=0}^{y=1} c\left(x^2+y^2+\tfrac{1}{3}\right) dy \right) dx.$$

Now we are left with an area integral that we know how to evaluate. Thus

$$M = \int_{x=0}^{x=1} c\left[x^2y + \tfrac{1}{3}y^3 + \tfrac{1}{3}y\right]_{y=0}^{y=1} dx$$
$$= \int_{x=0}^{x=1} c\left(\tfrac{2}{3}+x^2\right) dx$$
$$= c\left[\tfrac{2}{3}x + \tfrac{1}{3}x^3\right]_{x=0}^{x=1} = c.$$

The main point to notice is that after each integration, the number of variables is reduced by one.

Exercise 18

The density of the material in a block B with faces $x = 0$, $x = 2$, $y = 1$, $y = 2$, $z = 2$, $z = 5$ is given by $f(x, y, z) = x + y + z$.

Find the mass of the block.

Exercise 19

The density of a rectangular block B bounded by the planes $x = 1$, $x = 2$, $y = 0$, $y = 3$, $z = -1$, $z = 0$ is given by $f(x, y, z) = x(y + 1) - z$.

Find the mass of the block.

Sometimes it is more appropriate to evaluate a volume integral using cylindrical coordinates (introduced in Subsection 4.1 of Unit 15). We must then divide the region of integration into volume elements with surfaces $\rho =$ constant, $\phi =$ constant and $z =$ constant. A typical volume element is shown in Figure 23. It is approximately a rectangular cuboid of sides δz, $\rho\,\delta\phi$ and $\delta\rho$, so its volume is $\delta V \simeq \rho\,\delta z\,\delta\phi\,\delta\rho$, and this approximation improves as δz, $\delta\phi$ and $\delta\rho$ tend to zero. Note the factor ρ again, coming from the fact that $\rho\,\delta\phi$ is an arc length.

If the region of integration is a cylinder of radius a and height h, with the base of the cylinder on the (x, y)-plane, then in cylindrical coordinates the limits of integration are $z = 0$, $z = h$, $\phi = -\pi$, $\phi = \pi$, $\rho = 0$, $\rho = a$. So the volume integral of a scalar field f over the cylinder is as follows.

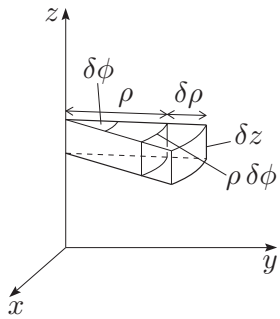


Figure 23 Volume element in cylindrical coordinates

Volume integral in cylindrical coordinates

$$\int_B f\,dV = \int_B f(\rho,\phi,z)\,\rho\,dz\,d\phi\,d\rho$$
$$= \int_{\rho=0}^{\rho=a}\left(\int_{\phi=-\pi}^{\phi=\pi}\left(\int_{z=0}^{z=h} f\rho\,dz\right)d\phi\right)d\rho. \qquad (20)$$

In most applications that you will meet in this module, the scalar field f is independent of the azimuthal angle ϕ, so the ϕ-integration simply gives a factor $\int_{-\pi}^{\pi} d\phi = 2\pi$, and equation (20) becomes

$$\int_B f\,dV = 2\pi\int_{\rho=0}^{\rho=a}\left(\int_{z=0}^{z=h} f\rho\,dz\right)d\rho, \qquad (21)$$

which is an area integral.

Example 10

The density of a cylinder of height h and radius a is given by $f = k\rho^2$, where k is a positive constant and ρ is the perpendicular distance from the axis of the cylinder.

Find the total mass of the cylinder when $h = 2$, $a = 0.5$ and $k = 4$.

Solution

Use cylindrical coordinates with the axis of the cylinder aligned along the z-axis and its base on the (x, y)-plane. The total mass of the cylinder is given by

$$M = \int_B f\,dV = \int_B f\rho\,dz\,d\phi\,d\rho = \int_B k\rho^3\,dz\,d\phi\,d\rho.$$

Using equation (21),

$$M = 2\pi\int_{\rho=0}^{\rho=a}\left(\int_{z=0}^{z=h} k\rho^3\,dz\right)d\rho = 2\pi k\int_0^a \rho^3 h\,d\rho = \tfrac{1}{2}\pi k a^4 h.$$

When $h = 2$, $a = 0.5$ and $k = 4$, the mass of the cylinder is $\frac{1}{4}\pi$ or about 0.79 kg.

Now we turn our attention to spherical coordinates (introduced in Subsection 4.2 of Unit 15). In order to evaluate volume integrals in this system, we need to find the volume δV of the volume element that is shown in Figure 24.

This volume is approximately a rectangular cuboid with sides PQ, PR and PS. Hence the volume of the element is

$$\delta V \simeq PQ \times PR \times PS.$$

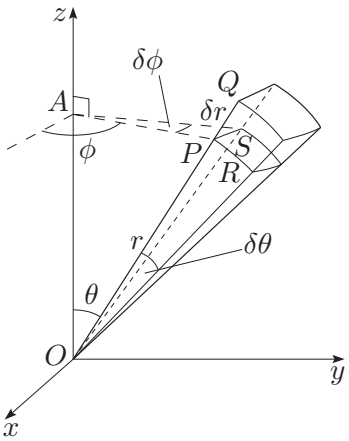


Figure 24 Volume element in spherical coordinates

From Figure 24 we have line $PQ = \delta r$, arc $PR = r\,\delta\theta$, and arc $PS = AP\,\delta\phi = (OP\sin\theta)\,\delta\phi = r\sin\theta\,\delta\phi$. So the volume element is

$$\delta V \simeq r^2 \sin\theta\,\delta\phi\,\delta\theta\,\delta r,$$

and this approximation improves as $\delta\phi$, $\delta\theta$ and δr tend to zero. Hence the volume integral of a scalar field f, over a region B, is given by the following.

Volume integral in spherical coordinates

$$\int_B f\,dV = \int_B f(r,\theta,\phi)\,r^2\sin\theta\,d\phi\,d\theta\,dr. \tag{22}$$

Once the function $f(r,\theta,\phi)$ is given, equation (22) can be evaluated as three successive integrals over ϕ, θ and r.

In many cases, the scalar field f is spherically symmetric so that f depends on r only, not on θ or ϕ. In such cases, and for a region contained between spherical shells of radii R_1 and R_2 ($R_1 < R_2$), equation (22) takes the form

$$\int_B f\,dV = \int_{r=R_1}^{r=R_2}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{\phi=-\pi}^{\phi=\pi} f(r)\,r^2\sin\theta\,d\phi\right)d\theta\right)dr.$$

The ϕ-integral gives a factor 2π, and the θ-integral gives $\left[-\cos\theta\right]_0^{\pi} = 2$. Thus in such cases,

$$\int_B f\,dV = 4\pi\int_{R_1}^{R_2} f(r)\,r^2\,dr. \tag{23}$$

Exercise 20

By evaluating a volume integral, find the mass of a sphere of radius R that is centred at the origin and whose density is given by $f = c + \alpha r$, where c and α are positive constants, and r is the distance from the origin.

In the examples and exercises that we have considered in this subsection, the regions of integration have been rectangular cuboids, cylinders and spheres. The boundaries of these regions coincide with coordinate surfaces in the appropriate coordinate system, so the limits of integration are all constants. In Subsection 3.3 we will look at examples where the regions of integration cut across coordinate surfaces, and as a result some of the limits of integration are functions. First, in the next subsection, we consider an application of volume integrals to the calculation of moments of inertia of rigid bodies.

3.2 Moments of inertia of rigid bodies

In Subsection 2.3 we calculated the moments of inertia of laminas in the (x, y)-plane, with mass distributions described by surface density functions $f(x, y)$. We now consider a rigid body that can rotate about the z-axis.

The mass distribution in a rigid body is modelled by a density function $f(x, y, z)$ with units of $\mathrm{kg\,m^{-3}}$. As in Subsection 2.3, our starting point is equation (12), giving the moment of inertia of a rigid body as the limit of a summation.

Suppose that a rigid body is divided into a very large number N of volume elements, modelled as particles. The mass of an element is approximately $f_i\,\delta V_i$, where δV_i is the volume of the element, and f_i is the density at a point inside the element. So the moment of inertia of the element about the z-axis is approximately $(f_i\,\delta V_i)d_i^2 = f_i d_i^2\,\delta V_i$, where d_i is the perpendicular distance of the ith element from the z-axis (see Figure 25). The moment of inertia of the whole body about the z-axis is found by summing over all elements and taking the limit as $N \to \infty$. The moment of inertia about the z-axis is then the volume integral of the function fd^2 over the region of space B occupied by the body.

The function $d(x, y, z)$ gives the distance of the point (x, y, z) from the point z on the z-axis, which is the perpendicular distance of the element from the z-axis.

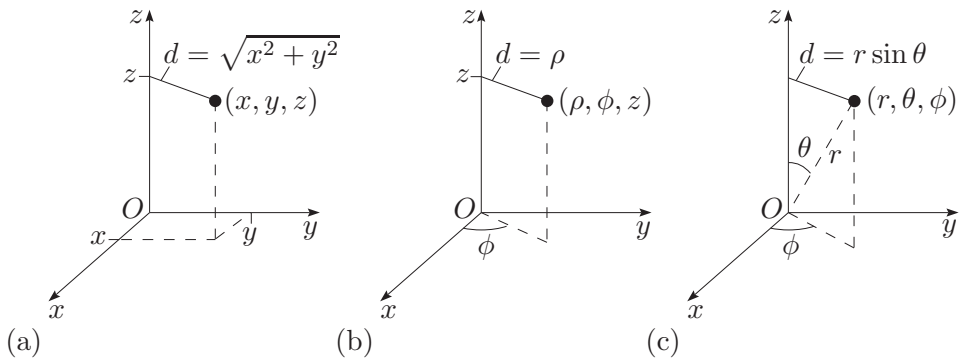


Figure 25 Perpendicular distance d from the z-axis in (a) Cartesian, (b) cylindrical and (c) spherical coordinate systems

Moment of inertia of a rigid body

The moment of inertia about the z-axis of a rigid body with density function f occupying a region B of space is

$$I = \int_B fd^2\,dV, \tag{24}$$

where d is the perpendicular distance from the z-axis, so

$$fd^2 = \begin{cases} f(x, y, z)\,(x^2 + y^2) & \text{in Cartesian coordinates,} \\ f(\rho, \phi, z)\,\rho^2 & \text{in cylindrical coordinates,} \\ f(r, \theta, \phi)\,(r \sin\theta)^2 & \text{in spherical coordinates.} \end{cases} \tag{25}$$

Example 11

Starting from equation (24), evaluate the moment of inertia I of a uniform sphere of radius R and constant density D about an axis through its centre.

Solution

Since the region of integration B is a sphere, we use spherical coordinates.

Let the centre of the sphere be at the origin, and we will calculate the moment of inertia about the z-axis. The density function $f = D$ is constant, and the distance d from the z-axis in spherical coordinates is $d = r\sin\theta$ (see Figure 25(c)). The moment of inertia of the sphere is

$$\begin{aligned} I &= \int_{\text{sphere}} D(r\sin\theta)^2\,dV \\ &= D\int_{\text{sphere}} (r\sin\theta)^2 r^2\sin\theta\,dr\,d\theta\,d\phi \\ &= D\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{\phi=-\pi}^{\phi=\pi} r^4\sin^3\theta\,d\phi\right)d\theta\right)dr. \end{aligned}$$

The ϕ-integral yields the factor

$$\int_{-\pi}^{\pi} d\phi = 2\pi,$$

the θ-integral yields the factor

$$\int_0^{\pi}\sin^3\theta\,d\theta = \int_0^{\pi}\sin\theta(1-\cos^2\theta)d\theta = \left[-\cos\theta + \tfrac{1}{3}\cos^3\theta\right]_0^{\pi} = \tfrac{4}{3},$$

and the r-integral gives

$$\int_0^{R} r^4\,dr = \tfrac{1}{5}R^5.$$

Hence the volume integral yields the moment of inertia

$$I = \tfrac{8}{15}\pi D R^5.$$

Note that in Example 11 the limits are all constant so the three integrals can be carried out in any order without changing the limits.

Exercise 21

Determine the mass M of a uniform sphere of radius R and density D by evaluating a volume integral, and hence show that the moment of inertia of the sphere about an axis through its centre is $I = \frac{2}{5}MR^2$.

Exercise 22

Show that the moment of inertia of a uniform solid cylinder of mass M, height h and radius a about its axis is $I = \frac{1}{2}Ma^2$.

Exercise 23

Find the moment of inertia of a uniform spherical shell of mass M, outer radius b and inner radius a about an axis through its centre.

The answer to Exercise 23 can be used to find the moment of inertia of a very thin spherical shell, such as a ping-pong ball, about an axis through its centre. We put $b = a + h$, where h, the thickness of the shell, is very small compared with a. Then we have

$$I_{\text{shell}} = \frac{2M(b^5 - a^5)}{5(b^3 - a^3)} = \frac{2M((a+h)^5 - a^5)}{5((a+h)^3 - a^3)} = \frac{2Ma^5\left(\left(1+\frac{h}{a}\right)^5 - 1\right)}{5a^3\left(\left(1+\frac{h}{a}\right)^3 - 1\right)}.$$

We can expand

$$\left(1+\frac{h}{a}\right)^5 \simeq 1 + \frac{5h}{a} \quad \text{and} \quad \left(1+\frac{h}{a}\right)^3 \simeq 1 + \frac{3h}{a}$$

See Unit 7 for Taylor polynomials.

by using a Taylor polynomial, keeping only the first-degree (i.e. linear) terms in h, since h/a is very much less than 1. Thus

$$I_{\text{thin shell}} \simeq \frac{2Ma^5(5h/a)}{5a^3(3h/a)} = \tfrac{2}{3}Ma^2.$$

The disc was assumed to be a lamina in the (x, y)-plane.

You may have noticed that the moment of inertia of the uniform cylinder in Exercise 22 is the same as the moment of inertia of the uniform disc in Exercise 15. This is not surprising since the moment of inertia is the sum over all elements of the quantity (mass) $\times$ (distance from z-axis)^2, which is independent of the distribution of mass in the z-direction. It follows that the moment of inertia of a uniform body is unchanged by projecting the mass distribution of the body onto the (x, y)-plane, provided that distances from the z-axis do not change.

The moment of inertia of any solid body about the z-axis can always, in principle, be evaluated as an area integral over a region of the (x, y)-plane, *provided* that the equivalent projected surface density in the (x, y)-plane is known. So, for example, the z-integration of the volume integral in the solution to Exercise 22 yields a factor Dh, the mass per unit volume times the height. This is equivalent to a mass per unit area – that is, a surface density – in the (x, y)-plane. The integrals over ρ and ϕ that are left after the z-integral is evaluated are the same as those that you evaluated in Exercise 15 to get the moment of inertia of a disc in the (x, y)-plane.

Exercise 24

Find the moment of inertia of a uniform cylindrical shell of mass M, outer radius b and inner radius a about its axis.

Exercise 25

Show that the moment of inertia of a uniform solid cylinder of mass M, length L and radius R, about an axis through the centre of the cylinder that is *normal* to the axis of the cylinder, is given by

$$I = \tfrac{1}{12}M(3R^2 + L^2).$$

(*Hint:* Use cylindrical coordinates with the z-axis aligned along the cylinder axis so that the perpendicular distance to the axis of rotation is *no longer* the perpendicular distance to the z-axis.)

Exercise 26

The density of a sphere of radius R varies as $f(\mathbf{r}) = c|\mathbf{r}|^2$, where c is a constant and $\mathbf{r}$ is measured from the sphere's centre.

(a) Determine the mass of the sphere.

(b) Find the moment of inertia of the sphere about an axis through its centre.

3.3 Volume integrals with irregular boundaries

Here we evaluate volume integrals where the boundaries of the region of integration do not coincide with coordinate surfaces. The limits on some of the integrals are then functions that must be determined. Thus the first stage in the evaluation of a volume integral is to decide on a coordinate system and the limits of integration. The second stage is to evaluate the resulting single integrals.

In this subsection we will illustrate how limits are decided on in cases where we use Cartesian coordinate systems for non-rectangular regions of integration. The procedure is to carry out the z-integration first. When this is done, we are left with an area integral in the (x, y)-plane, which we can evaluate as in Subsection 1.3.

So instead of the volume integral with constant limits as in equation (19), we have variable limits in the z-direction (and possibly the y-direction).

Volume integral over a general volume

$$\int_B f(x,y,z)\,dV = \int_{x=a}^{x=b}\left(\int_{y=\alpha(x)}^{y=\beta(x)}\left(\int_{z=\gamma(x,y)}^{z=\psi(x,y)} f(x,y,z)\,dz\right)dy\right)dx. \qquad (26)$$

To illustrate the method, we consider a volume integral of a function f over the cylindrical region of radius a shown in Figure 26(a). Note that this cylinder has its axis on the y-axis, not the z-axis, and the curved boundary does not coincide with any Cartesian coordinate surface. Of course, depending on the form of the integrand f, it is often more convenient to use a cylindrical coordinate system here, given the shape of the region. (This would require redefining the y-axis as the z-axis, and introducing the ρ and ϕ coordinates accordingly.) However, for the purpose of this illustration, we assume that the form of f is such that there is no disadvantage in using Cartesian coordinates.

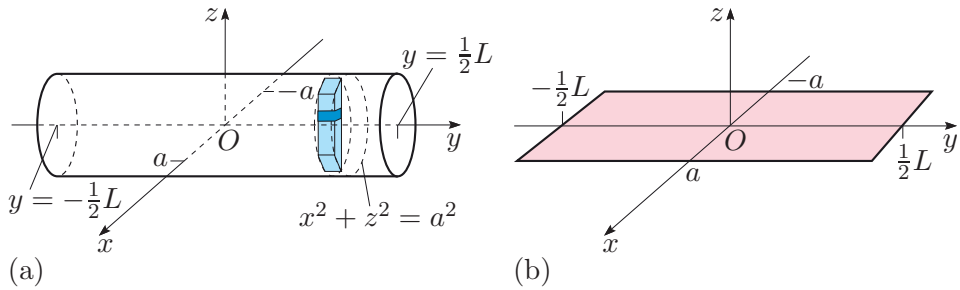


Figure 26 Cylindrical region of integration: (a) a vertical column; (b) projection of the cylinder onto the (x, y)-plane

Recall that the volume elements in Cartesian coordinates are small blocks of volume $\delta V = \delta z\, \delta y\, \delta x$. We choose to integrate first over z. The z-integration then represents summing up the volume elements in the vertical column in Figure 26(a). Where this column intersects the bottom and top curved boundaries of the cylinder are the lower and upper limits of the z-integration. In this case the limits are $z = -\sqrt{a^2 - x^2}$ and $z = \sqrt{a^2 - x^2}$. Hence the z-integral is

The curved boundary has the equation $x^2 + z^2 = a^2$ so is made up of $z = \sqrt{a^2 - x^2}$ $(z \geq 0)$ and $z = -\sqrt{a^2 - x^2}$ $(z < 0)$.

$$\int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} f(x, y, z)\, dz = h(x, y).$$

The result of the integration over z is $h(x, y)$, some function of x and y only. To complete the volume integral, we must evaluate the area integral

$$\int_S h(x, y)\, dy\, dx,$$

where the region S is the projection of the cylinder onto the (x, y)-plane. This projection is just the rectangle shown in Figure 26(b). The following example illustrates how this method works.

Example 12

Use Cartesian coordinates to evaluate the volume integral of the function $x^2y^2(a^2 - x^2)^{3/2}$ over the cylinder of radius a, length L, centred at the origin and with its axis along the y-axis (i.e. as shown in Figure 26(a)).

Solution

Figure 26(a) shows the cylinder lying with its centre at the origin and its axis on the y-axis. With $f(x, y, z) = x^2y^2(a^2 - x^2)^{3/2}$, we require

$$\int_{\text{cylinder}} f(x, y, z)\, dV = \int_{\text{cylinder}} x^2y^2(a^2 - x^2)^{3/2}\, dz\, dy\, dx.$$

The limits of the z-integral are chosen as described above, so the z-integral gives

$$\begin{aligned} h(x, y) &= \int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} x^2y^2(a^2 - x^2)^{3/2}\, dz \\ &= x^2y^2(a^2 - x^2)^{3/2} \int_{z=-\sqrt{a^2-x^2}}^{z=\sqrt{a^2-x^2}} 1\, dz \\ &= 2x^2y^2(a^2 - x^2)^2. \end{aligned}$$

We are now left with the x- and y-integrations in equation (26). We must evaluate the area integral of $h(x, y)$ over the projection of the cylinder on the (x, y)-plane, which is the rectangle shown in Figure 26(b). Following Procedure 1 for evaluating area integrals, we first note that the limits of the x-integral are $-a$ and a, and the limits on the y-integral are $-\frac{1}{2}L$ and $\frac{1}{2}L$. Thus we have

$$\begin{aligned} \int_{\text{cylinder}} f(x, y, z)\, dV &= \int_{\text{rectangle}} h(x, y)\, dy\, dx \\ &= \int_{x=-a}^{x=a} \left(\int_{y=-L/2}^{y=L/2} 2x^2y^2(a^2 - x^2)^2\, dy \right) dx \\ &= 2 \int_{x=-a}^{x=a} x^2(a^2 - x^2)^2 \left[\tfrac{1}{3}y^3 \right]_{y=-L/2}^{y=L/2} dx \\ &= \tfrac{1}{6}L^3 \left[\tfrac{1}{3}a^4x^3 - \tfrac{2}{5}a^2x^5 + \tfrac{1}{7}x^7 \right]_{-a}^{a} \\ &= \tfrac{8}{315}L^3a^7. \end{aligned}$$

Finally, it should be noted that had we chosen a cylindrical coordinate system to perform this volume integral, we would have been left with much harder single integrals to evaluate than those given above. So in this case, as a result of the form of the integrand $f(x, y, z)$, we really are justified in using Cartesian coordinates.

The steps that we have used to evaluate a volume integral are summarised in the following procedure.

Procedure 2 Evaluating volume integrals

To evaluate a volume integral

$$\int_B f(x, y, z)\, dV,$$

carry out the following steps.

1. Draw two diagrams, showing the region of integration B with the equations of the upper and lower boundaries marked, and the projection S of this region onto the (x, y)-plane.
2. Within the region B, draw a column perpendicular to the (x, y)-plane and determine the limits of the z-integration, $z = \gamma(x, y)$ and $z = \psi(x, y)$, say.
3. Evaluate the single integral of $f(x, y, z)$ over z between $z = \gamma(x, y)$ and $z = \psi(x, y)$, keeping x and y constant, to find the function $h(x, y)$ defined by
$$h(x, y) = \int_{z=\gamma(x,y)}^{z=\psi(x,y)} f(x, y, z)\, dz.$$
4. Evaluate the area integral of $h(x, y)$ over the region S using Procedure 1.

In Example 12 we had $\gamma(x, y) = -\sqrt{a^2 - x^2}$ and $\psi(x, y) = \sqrt{a^2 - x^2}$.

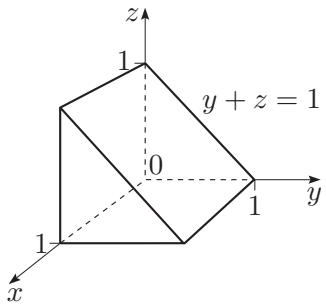


Figure 27

Exercise 27

Find the value of the volume integral of the function

$$f(x, y, z) = x^2yz$$

over the wedge-shaped region bounded by the planes $z = 0$, $y = 0$, $x = 0$, $x = 1$ and $y + z = 1$ (as shown in Figure 27).

Exercise 28

Find the value of the volume integral of the function

$$f(x, y, z) = z + 3x - 2$$

over the region bounded by the parabolic surfaces $y = 1 - x^2$ and $y = x^2 - 1$, and the planes $z = 0$ and $z = 1$.

One often encounters volume integrals over regions with boundaries that do not coincide with coordinate surfaces but which are best evaluated using cylindrical or spherical coordinates as opposed to Cartesian coordinates. To illustrate how this is done, we now determine the moment of inertia of a solid cone about its axis.

Example 13

Find the moment of inertia of a solid uniform cone of mass M, height h and base radius a about its axis (see Figure 28).

(*Hint*: Use cylindrical coordinates and integrate first over z; express the height z of the vertical column in terms of a and h.)

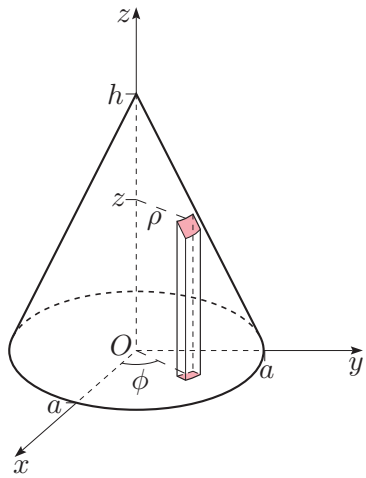


Figure 28 A cone

Solution

Let the base of the cone lie in the (x, y)-plane with the axis of the cone on the z-axis, as shown in Figure 28. Use cylindrical coordinates and integrate over z first. This represents first summing elements in the column starting at the base, $z = 0$, and ending at $z = h(1 - \rho/a)$, the top of the column. Thus taking the uniform density to be D, the moment of inertia of the cone is

$$I_{\text{cone}} = \int_{\text{cone}} D\rho^2 \, dV$$
$$= D \int_{\rho=0}^{\rho=a} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=0}^{z=h(1-\rho/a)} \rho^3 \, dz \right) d\phi \right) d\rho.$$

Integrating over z leaves the area integral over the circular base in the (x, y)-plane, which using equation (10) gives

$$I_{\text{cone}} = 2\pi D \int_0^a \rho^3 h(1 - \rho/a) \, d\rho = 2\pi Dh \left[\tfrac{1}{4}\rho^4 - \tfrac{1}{5}\rho^5/a \right]_0^a = \tfrac{1}{10}\pi Dha^4.$$

The mass of the cone is simply the volume integral of D over the region occupied by the cone, which is

$$M = D \int_{\rho=0}^{\rho=a} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=0}^{z=h(1-\rho/a)} \rho \, dz \right) d\phi \right) d\rho$$
$$= 2\pi D \int_0^a \rho h(1 - \rho/a) \, d\rho = 2\pi Dh \left[\tfrac{1}{2}\rho^2 - \tfrac{1}{3}\rho^3/a \right]_0^a = \tfrac{1}{3}D\pi a^2 h.$$

Hence

$$I_{\text{cone}} = \tfrac{3}{10}Ma^2.$$

Exercise 29

Consider a region B of space bounded by the circular paraboloid $z/h = (x^2 + y^2)/a^2$ $(h > 0)$ and the plane $z = h$, as shown in Figure 29.

(a) Determine the volume of region B.

(b) If region B consists of a uniform solid of mass M, show that its moment of inertia about the z-axis is given by $I = \frac{1}{3}Ma^2$.

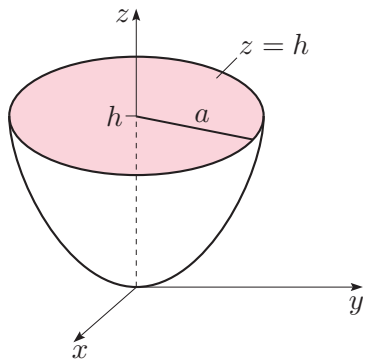


Figure 29 A circular paraboloid

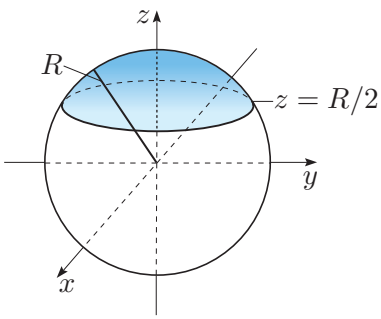


Figure 30 A spherical cap

Exercise 30

Consider a uniform solid of mass M whose shape is given by the interior of a *spherical cap* defined by the region of space described by the inequalities $x^2 + y^2 + z^2 \leq R^2$ and $z \geq R/2 > 0$ (see Figure 30). By using cylindrical coordinates, do the following.

(a) Show that the volume of the solid is $\frac{5}{24}\pi R^3$.

(b) Show that the moment of inertia of the solid about its axis of symmetry is $\frac{53}{200}MR^2$.

(*Hint*: You may use the indefinite integral

$$\int x^3\sqrt{a^2 - x^2}\,dx = \tfrac{1}{5}(a^2 - x^2)^{5/2} - \tfrac{1}{3}a^2(a^2 - x^2)^{3/2}.)$$

4 Area of a surface

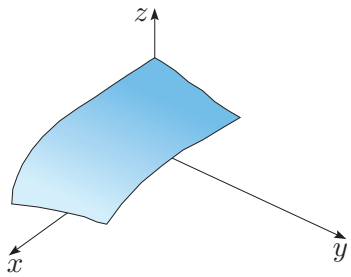


Figure 31 Portion of the surface $z = f(x, y)$

We use Σ to denote the *surface area* of the *curved surface*. This upper-case sigma should not be confused with the summation symbol $\sum$; context should make the distinction clear.

Note that this approach and the notation used differs somewhat from the derivation of area integrals in Subsection 1.2.

In Unit 16 we studied the scalar line integral, and showed that as a particular application it can be used to find the length of a curve. In this section we find a corresponding application of the area integrals studied in Section 1; this enables us to calculate the area of a curved surface.

In Section 1 we defined and calculated many area integrals. However, the areas that we considered were always in a plane. Sometimes it is useful to be able to calculate areas of surfaces in three-dimensional space – for example, we may need to know the area of a curved surface like part of a sphere. Fortunately, there is a way to find such an area in terms of area integrals.

Consider a surface in space defined by the equation $z = f(x, y)$, a portion of which is shown in Figure 31.

Suppose that we wish to calculate the area Σ of that part of the surface that lies above the rectangle defined by $a \leq x \leq b$ and $c \leq y \leq d$. Take a small element of the area above the point (x_0, y_0), as shown in Figure 32.

If small enough, the element will be approximately rectangular with vertices

$$P = (x_0, y_0, f(x_0, y_0)),$$
$$Q = (x_0 + \delta x, y_0, f(x_0 + \delta x, y_0)),$$
$$R = (x_0, y_0 + \delta y, f(x_0, y_0 + \delta y)),$$
$$S = (x_0 + \delta x, y_0 + \delta y, f(x_0 + \delta x, y_0 + \delta y)).$$

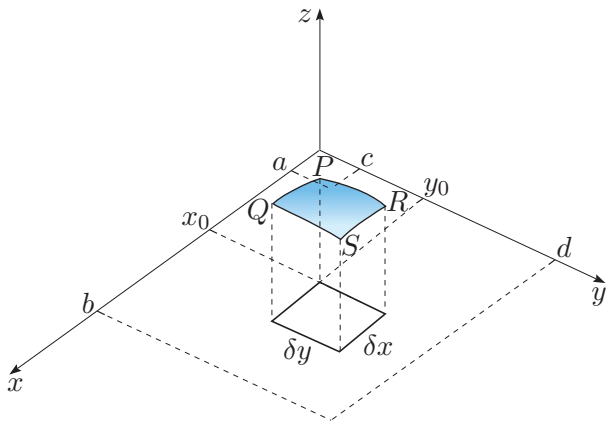


Figure 32 Small element of the surface $z = f(x, y)$

Now recall that a point (x, y, z) in space can be represented as a position vector $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. We can therefore think of the edges PQ and PR as forming two vectors, $\mathbf{v}_1$ and $\mathbf{v}_2$. We have

$$\begin{aligned}\mathbf{v}_1 &= \overrightarrow{PQ} \\ &= \big((x_0 + \delta x)\,\mathbf{i} + y_0\,\mathbf{j} + f(x_0 + \delta x, y_0)\,\mathbf{k}\big) - \big(x_0\,\mathbf{i} + y_0\,\mathbf{j} + f(x_0, y_0)\,\mathbf{k}\big) \\ &= \delta x\,\mathbf{i} + \big(f(x_0 + \delta x, y_0) - f(x_0, y_0)\big)\,\mathbf{k} \\ &\simeq \delta x\,\mathbf{i} + \delta x\,\frac{\partial f}{\partial x}\,\mathbf{k},\end{aligned}$$

where we have used the approximation

$$\frac{\partial f}{\partial x} \simeq \frac{f(x_0 + \delta x, y_0) - f(x_0, y_0)}{\delta x}.$$

Similarly,

$$\begin{aligned}\mathbf{v}_2 &= \overrightarrow{PR} \\ &= \big(x_0\,\mathbf{i} + (y_0 + \delta y)\,\mathbf{j} + f(x_0, y_0 + \delta y)\,\mathbf{k}\big) - \big(x_0\,\mathbf{i} + y_0\,\mathbf{j} + f(x_0, y_0)\,\mathbf{k}\big) \\ &= \delta y\,\mathbf{j} + \big(f(x_0, y_0 + \delta y) - f(x_0, y_0)\big)\,\mathbf{k} \\ &\simeq \delta y\,\mathbf{j} + \delta y\,\frac{\partial f}{\partial y}\,\mathbf{k}.\end{aligned}$$

Now the area of the parallelogram formed by two vectors is given by the magnitude of their cross product. In our context this becomes

This was covered in Unit 2, Subsection 2.3.

$$\begin{aligned}\delta\Sigma &= |\mathbf{v}_1 \times \mathbf{v}_2| \\ &\simeq \left|\left(\delta x\,\mathbf{i} + \delta x\,\frac{\partial f}{\partial x}\,\mathbf{k}\right) \times \left(\delta y\,\mathbf{j} + \delta y\,\frac{\partial f}{\partial y}\,\mathbf{k}\right)\right| \\ &= \left|-\delta x\,\delta y\,\frac{\partial f}{\partial x}\,\mathbf{i} - \delta x\,\delta y\,\frac{\partial f}{\partial y}\,\mathbf{j} + \delta x\,\delta y\,\mathbf{k}\right| \\ &= \delta x\,\delta y\,\sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2},\end{aligned}$$

where $\delta\Sigma$ is the area of the small element.

The area Σ of the whole surface is given approximately by adding up the areas of all the small elements, so

$$\Sigma \simeq \sum_{\text{small elements}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, \delta y\, \delta x.$$

As the size of the elements tends to zero, we can pass to the limit, and the sum becomes an area integral in the sense of Section 1:

$$\Sigma = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy \right) dx.$$

As in Section 1 the integrations can be performed in either order.

We have dealt with a portion of surface lying over a rectangular region of the plane, but as in Section 1 this can be extended to more general regions (see Example 15). The limits of the two integrals are then determined by the context, but often, as we found in Section 1, the limits of the inner integral will be functions rather than constants.

Procedure 3 Finding the area of a surface

Suppose that a surface is described in the form $z = f(x, y)$. To find the area Σ of a portion of the surface lying over the region S in the (x, y)-plane, carry out the following steps.

1. Form the function
$$g(x, y) = \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}.$$
2. Calculate the area integral
$$\Sigma = \int_S g(x, y)\, dA,$$
as in Procedure 1.

The idea is simple enough, although the integrals can be extremely messy – even for fairly simple surfaces.

Example 14

Express the area of the surface $z = \frac{1}{2}x^2 + \frac{1}{2}y^2$, lying over the square defined by $-1 \leq x \leq 1$, $-1 \leq y \leq 1$, as an area integral. (Do *not* attempt to evaluate the integral.)

Solution

With $f(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2$, we have

$$\frac{\partial f}{\partial x} = x \quad \text{and} \quad \frac{\partial f}{\partial y} = y.$$

Thus, following Procedure 3, we write

$$g(x, y) = \sqrt{1 + x^2 + y^2},$$

and obtain the integral

$$\Sigma = \int_S g(x, y)\, dA.$$

With the appropriate limits this becomes

$$\Sigma = \int_{x=-1}^{x=1} \left(\int_{y=-1}^{y=1} \sqrt{1 + x^2 + y^2}\, dy \right) dx.$$

Note that this integral cannot easily be evaluated by analytical means, even with the assistance of a computer algebra package. However, a numerical computation gives it a value of about 5.123.

As in Section 2, the evaluation of a surface integral may sometimes be dramatically simplified by using polar coordinates.

Example 15

Find the area of the portion of a sphere of radius a, centred at the origin, lying above the (x, y)-plane and defined by the angle ψ as shown in Figure 33 (where $0 \leq \psi \leq \pi/2$). (Recall that the volume and moment of inertia of a spherical cap were calculated in Exercise 30.)

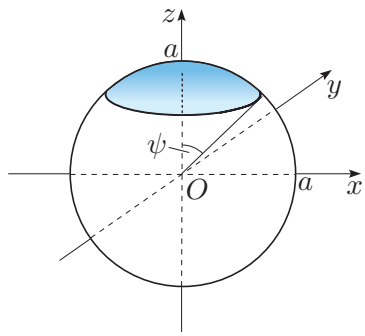


Figure 33 Spherical cap defined in spherical coordinates by $r = a$ and $0 \leq \theta \leq \psi$

Solution

This surface is given by the equation

$$f(x, y) = \sqrt{a^2 - x^2 - y^2},$$

since $x^2 + y^2 + z^2 = a^2$, and the corresponding region in the (x, y)-plane is a circular disc of radius $a \sin \psi$. Now

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{a^2 - x^2 - y^2}} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{a^2 - x^2 - y^2}},$$

and we must compute

$$\Sigma = \int_{x=a}^{x=b} \left(\int_{y=\alpha(x)}^{y=\beta(x)} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy \right) dx$$

over the disc in the (x, y)-plane given by

$$x^2 + y^2 \leq a^2 \sin^2 \psi.$$

Compare with Figure 15.

Following the discussion after Exercise 10, we see that the limits are

$$-a\sin\psi \le x \le a\sin\psi, \quad -\sqrt{a^2\sin^2\psi - x^2} \le y \le \sqrt{a^2\sin^2\psi - x^2},$$

and we must evaluate

$$\Sigma = \int_{x=-a\sin\psi}^{x=a\sin\psi} \left(\int_{y=-\sqrt{a^2\sin^2\psi - x^2}}^{y=\sqrt{a^2\sin^2\psi - x^2}} \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dy \right) dx$$

$$= \int_{x=-a\sin\psi}^{x=a\sin\psi} \left(\int_{y=-\sqrt{a^2\sin^2\psi - x^2}}^{y=\sqrt{a^2\sin^2\psi - x^2}} \frac{a}{\sqrt{a^2 - x^2 - y^2}}\, dy \right) dx.$$

It is clear that this will be considerably easier to deal with if we use polar coordinates, so we proceed as in Section 2.

The function to be integrated is simply

$$g(r,\theta) = \frac{a}{\sqrt{a^2 - r^2}}.$$

Note that for the r-integral we use the trick given in equation (6), with $k = -\frac{1}{2}$, $n = -\frac{1}{2}$ and $f(r) = a^2 - r^2$.

The limits for the r-integration are 0 and $a\sin\psi$, while for θ the limits are simply $-\pi$ and π. Therefore the required integral is

$$\Sigma = \int_{r=0}^{r=a\sin\psi} \left(\int_{\theta=-\pi}^{\theta=\pi} \frac{a}{\sqrt{a^2 - r^2}}\, r\, d\theta \right) dr$$

$$= 2\pi a \int_{r=0}^{r=a\sin\psi} \frac{r}{\sqrt{a^2 - r^2}}\, dr$$

$$= 2\pi a \left[-\sqrt{a^2 - r^2} \right]_0^{a\sin\psi}$$

$$= 2\pi a \left(-a\cos\psi - (-a) \right)$$

$$= 2\pi a^2 (1 - \cos\psi).$$

We can confirm that this is a plausible answer by considering some specific values of ψ. When $\psi = 0$ we obtain 0, and when $\psi = \pi/2$ we obtain $2\pi a^2$ (which is the correct formula for the area of a hemisphere of radius a).

Exercise 31

Show that the area of the surface $z = \frac{1}{2}(x^2 + y^2)$, lying over a disc centred at the origin with radius a, is given by

$$\Sigma = \tfrac{2}{3}\pi\big((1 + a^2)^{3/2} - 1\big).$$

Exercise 32

Find the surface area of the curved surface of a cone of height h and base radius a.

Learning outcomes

After studying this unit, you should be able to:

- evaluate area integrals over regions of the (x, y)-plane using Cartesian or polar coordinate systems
- use area integrals for evaluating population models in the (x, y)-plane, and for obtaining the masses and moments of inertia of laminas
- evaluate volume integrals over regions of space using Cartesian, cylindrical or spherical coordinate systems
- use volume integrals to evaluate the masses and moments of inertia of solids
- use area integrals to find surface areas.

Solutions to exercises

Solution to Exercise 1

$$\begin{aligned}\int_S x^2y^3\,dA &= \int_{x=0}^{x=2}\left(\int_{y=1}^{y=3} x^2y^3\,dy\right)dx\\ &= \int_{x=0}^{x=2}\left(\left[x^2(\tfrac{1}{4}y^4)\right]_{y=1}^{y=3}\right)dx\\ &= \int_{x=0}^{x=2}(20x^2)\,dx\\ &= \left[\tfrac{20}{3}x^3\right]_{x=0}^{x=2}\\ &= \tfrac{160}{3}.\end{aligned}$$

Solution to Exercise 2

We must evaluate the expression

$$\int_{y=1}^{y=2}\left(\int_{x=0}^{x=3} xy\,dx\right)dy.$$

Taking the integral over x first, treating y as a constant, we have

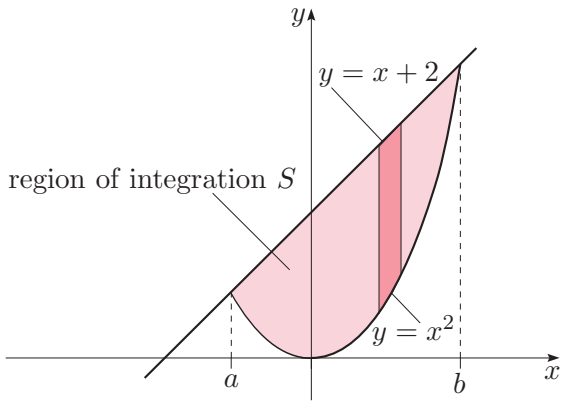
$$\int_{x=0}^{x=3} xy\,dx = y\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=3} = \tfrac{9}{2}y.$$

Now the y-integration gives

$$\int_{y=1}^{y=2}\tfrac{9}{2}y\,dy = \tfrac{9}{2}\left[\tfrac{1}{2}y^2\right]_{y=1}^{y=2} = \tfrac{27}{4}.$$

Solution to Exercise 3

Step 1: The diagram is shown below.



Step 2: To find the y-limits, we draw in a vertical strip at some general position in the region, as shown in the figure above. The lower point of this strip is then $y = x^2$, and the upper limit is $y = x + 2$. Note that for each vertical strip, the lower and upper vertical limits are on the parabola and straight line, respectively.

Step 3: The limits of the x-integration are $x = a$ and $x = b$, where these values of x are solutions of the simultaneous equations $y = x^2$ and $y = x + 2$, that is, of $x^2 - x - 2 = (x - 2)(x + 1) = 0$, giving $x = 2$ and $x = -1$. So $a = -1$ and $b = 2$.

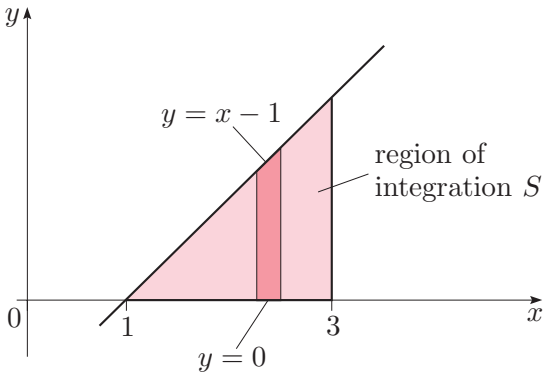
Step 4: $\displaystyle\int_S y\,dA = \int_{x=-1}^{x=2} \left(\int_{y=x^2}^{y=x+2} y\,dy \right) dx.$

Step 5: $\displaystyle\int_{y=x^2}^{y=x+2} y\,dy = \left[\tfrac{1}{2}y^2\right]_{y=x^2}^{y=x+2} = \tfrac{1}{2}(x+2)^2 - \tfrac{1}{2}x^4.$

Step 6: $\displaystyle\int_{x=-1}^{x=2} \left(\tfrac{1}{2}(x+2)^2 - \tfrac{1}{2}x^4\right) dx = \left[\tfrac{1}{6}(x+2)^3 - \tfrac{1}{10}x^5\right]_{x=-1}^{x=2} = \tfrac{36}{5}.$

Solution to Exercise 4

Step 1: The diagram is shown below.



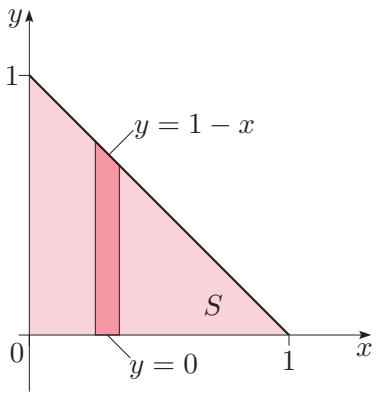
Step 2: The y-limits are $y = 0$ and $y = x - 1$.

Step 3: The x-limits are $x = 1$ and $x = 3$.

Step 4: $\displaystyle\int_S (x - y)\,dA = \int_{x=1}^{x=3} \left(\int_{y=0}^{y=x-1} (x - y)\,dy \right) dx.$

Step 5: $\displaystyle\int_{y=0}^{y=x-1} (x - y)\,dy = \left[-\tfrac{1}{2}(x-y)^2\right]_{y=0}^{y=x-1} = \tfrac{1}{2}x^2 - \tfrac{1}{2}.$

Step 6: $\displaystyle\int_{x=1}^{x=3} \left(\tfrac{1}{2}x^2 - \tfrac{1}{2}\right) dx = \left[\tfrac{1}{6}x^3 - \tfrac{1}{2}x\right]_{x=1}^{x=3} = \tfrac{10}{3}.$



Solution to Exercise 5

A diagram showing the region of integration S and a strip for the y-integration is shown in the margin.

Combining the steps of Procedure 1 gives

$$\begin{aligned}\int_S f\,dA &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1-x}(x+y)\,dy\right)dx\\ &= \int_{x=0}^{x=1}\left[\tfrac{1}{2}(x+y)^2\right]_{y=0}^{y=1-x}dx\\ &= \int_0^1\left(\tfrac{1}{2}-\tfrac{1}{2}x^2\right)dx\\ &= \left[\tfrac{1}{2}x-\tfrac{1}{6}x^3\right]_0^1 = \tfrac{1}{3}.\end{aligned}$$

Note that we can write integration limits as simple constants once we reach the final variable.

Solution to Exercise 6

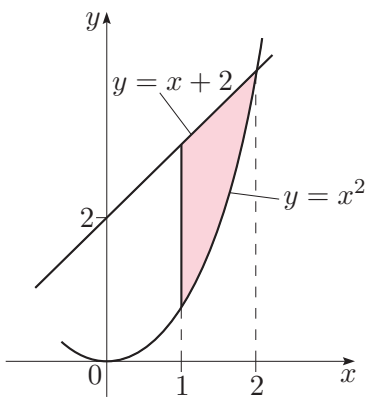
We have

$$\begin{aligned}\int_S f\,dA &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=\sqrt{1-x^2}}x\,dy\right)dx\\ &= \int_{x=0}^{x=1}\left[xy\right]_{y=0}^{y=\sqrt{1-x^2}}dx\\ &= \int_0^1 x\sqrt{1-x^2}\,dx\\ &= \left[-\tfrac{1}{3}(1-x^2)^{3/2}\right]_0^1 = \tfrac{1}{3}.\end{aligned}$$

For the x-integral, we have used the trick given in equation (6), with $k=-\frac{1}{2}$, $n=\frac{1}{2}$ and $f(x)=1-x^2$.

Solution to Exercise 7

The region is enclosed by the straight lines $x=1$, $x=2$, $y=x+2$ and the parabola $y=x^2$, as shown in the diagram below.

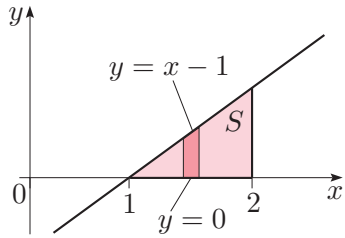


So we have

$$\int_{x=1}^{x=2}\left(\int_{y=x^2}^{y=x+2} 1\,dy\right)dx = \int_{x=1}^{x=2}\big[y\big]_{y=x^2}^{y=x+2}\,dx$$
$$= \int_1^2 (x+2-x^2)\,dx$$
$$= \big[\tfrac{1}{2}x^2 + 2x - \tfrac{1}{3}x^3\big]_1^2$$
$$= \tfrac{7}{6}.$$

Solution to Exercise 8

(a) Step 1: If we integrate over y first, then we *fix* x and draw in a *vertical* strip as shown in the figure in the margin.



Step 2: The y-limits are $y = 0$ and $y = x - 1$.

Step 3: The x-limits are $x = 1$ and $x = 2$.

Step 4: $\displaystyle\int_S (x^2+y^2)\,dA = \int_{x=1}^{x=2}\left(\int_{y=0}^{y=x-1}(x^2+y^2)\,dy\right)dx.$

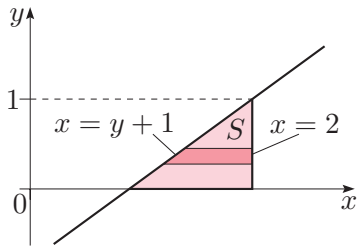
Step 5: $\displaystyle\int_{y=0}^{y=x-1}(x^2+y^2)\,dy = \big[x^2y + \tfrac{1}{3}y^3\big]_{y=0}^{y=x-1}$

$= x^2(x-1) + \tfrac{1}{3}(x-1)^3.$

Step 6: $\displaystyle\int_{x=1}^{x=2}\big(x^3 - x^2 + \tfrac{1}{3}(x-1)^3\big)\,dx$

$= \big[\tfrac{1}{4}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{12}(x-1)^4\big]_{x=1}^{x=2} = \tfrac{3}{2}.$

(b) Step 1: If we integrate over x first, then we *fix* y and draw in a *horizontal* strip as shown in the figure in the margin.



Step 2: The x-limits are $x = y + 1$ and $x = 2$.

Step 3: The y-limits are $y = 0$ and $y = 1$.

Step 4: $\displaystyle\int_S (x^2+y^2)\,dA = \int_{y=0}^{y=1}\left(\int_{x=y+1}^{x=2}(x^2+y^2)\,dx\right)dy.$

Step 5: $\displaystyle\int_{x=y+1}^{x=2}(x^2+y^2)\,dx = \big[\tfrac{1}{3}x^3 + xy^2\big]_{x=y+1}^{x=2}$

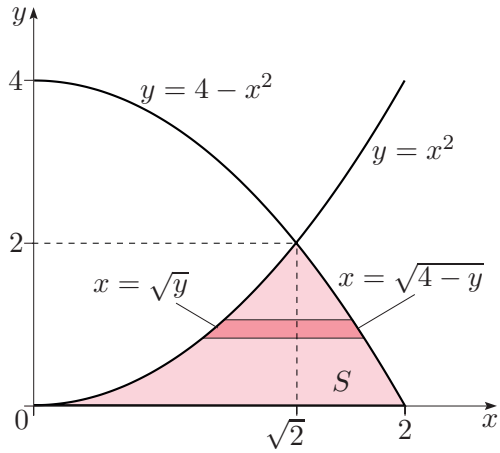
$= \tfrac{8}{3} + 2y^2 - \tfrac{1}{3}(y+1)^3 - (y+1)y^2.$

Step 6: $\displaystyle\int_{y=0}^{y=1}\big(\tfrac{8}{3} - \tfrac{1}{3}(y+1)^3 + y^2 - y^3\big)\,dy$

$= \big[\tfrac{8}{3}y - \tfrac{1}{12}(y+1)^4 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4\big]_{y=0}^{y=1} = \tfrac{3}{2}.$

Solution to Exercise 9

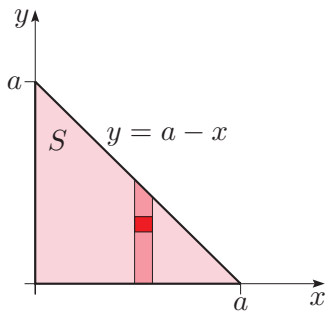
The region of integration S is shown shaded in the following figure.



The upper limit of the y-integral (outer integral) is given by the intersection of the two parabolas, where $y = x^2 = 4 - x^2$, that is, $x^2 = 2$. So the point of intersection is at $(\sqrt{2}, 2)$, as shown in the figure. Thus the area integral is

$$\begin{aligned}\int_S f\,dA &= \int_{y=0}^{y=2} \left(\int_{x=\sqrt{y}}^{x=\sqrt{4-y}} x\,dx \right) dy \\ &= \int_{y=0}^{y=2} \left[\tfrac{1}{2}x^2\right]_{\sqrt{y}}^{\sqrt{4-y}}\,dy \\ &= \int_0^2 (2-y)\,dy = \left[2y - \tfrac{1}{2}y^2\right]_0^2 = 2.\end{aligned}$$

Solution to Exercise 10



The total population is the area integral of the surface density function over the triangular region of the plate shown in the figure in the margin.

Integrating first over y, that is, using vertical strips, the lower and upper limits of the y-integral are $y = 0$ and $y = a - x$. The strips begin at $x = 0$ and end at $x = a$. Hence the total number of bacteria is given by the area integral

$$\begin{aligned}&A \int_{x=0}^{x=a} \left(\int_{y=0}^{y=a-x} \left(2 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right) dy \right) dx \\ &= A \int_{x=0}^{x=a} \left[2y - \frac{x^2 y}{a^2} - \frac{y^3}{3a^2} \right]_{y=0}^{y=a-x} dx \\ &= A \int_0^a \left(2(a-x) - \frac{x^2(a-x)}{a^2} - \frac{(a-x)^3}{3a^2} \right) dx \\ &= A \left[2ax - x^2 - \frac{x^3}{3a} + \frac{x^4}{4a^2} + \frac{(a-x)^4}{12a^2} \right]_0^a \\ &= \tfrac{5}{6} A a^2.\end{aligned}$$

When $A = 10^{12}$ and $a = 0.01$, the number of bacteria is 8.3×10^7.

Solution to Exercise 11

The mass of the lens is

$$\int_{\text{disc}} f\,dA = 2\pi k \int_{r=0}^{r=a} \left(1 - \frac{r^2}{a^2}\right) r\,dr$$
$$= 2\pi k \left[\frac{r^2}{2} - \frac{r^4}{4a^2}\right]_0^a = \tfrac{1}{2}\pi k a^2 = 2.9 \times 10^{-3}$$

for $a = 2.5 \times 10^{-2}$ and $k = 3$.

Solution to Exercise 12

(a) We have

$$J^2 = \int_{-\infty}^{\infty} \exp(-x^2)\,dx \times \int_{-\infty}^{\infty} \exp(-y^2)\,dy$$
$$= \int_{x=-\infty}^{x=\infty} \left(\int_{y=-\infty}^{y=\infty} \exp\left(-(x^2+y^2)\right) dy\right) dx,$$

showing that J^2 is an area integral of $\exp\left(-(x^2+y^2)\right)$ over the whole of the (x, y)-plane.

(b) In polar coordinates, $\exp\left(-(x^2+y^2)\right) = \exp(-r^2)$ and the whole of the (x, y)-plane can be regarded as a disc of infinite radius, denoted D. Thus

$$J^2 = \int_D \exp(-r^2)\,dA$$
$$= 2\pi \int_0^{\infty} \exp(-r^2)\,r\,dr$$
$$= \pi \int_0^{\infty} e^{-u}\,du = \pi\left[-e^{-u}\right]_0^{\infty} = \pi,$$

where the substitution $u = r^2$ was used in the evaluation of the r-integral. Hence $J = \sqrt{\pi}$.

Solution to Exercise 13

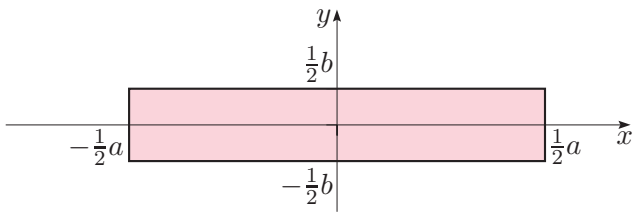
The required mass is the area integral of f on a disc of radius b. Noting that f depends on r only, we have

$$\int_S f\,dA = 2\pi A \int_0^b \exp(-r^2/a^2) r\,dr$$
$$= -\pi A a^2 \left[\exp(-r^2/a^2)\right]_0^b$$
$$= \pi a^2 A(1 - \exp(-b^2/a^2))$$
$$= \pi \times (1.5 \times 10^{-2})^2 \times 10^{-5}(1 - \exp(-1/1.5^2))$$
$$= 2.5 \times 10^{-9}$$

for $A = 10^{-5}$, $a = 1.5 \times 10^{-2}$ and $b = 0.01$.

Solution to Exercise 14

The ruler is modelled as a lamina in the (x, y)-plane with its centre at the origin. The region of integration is shown in the figure below.



Using Cartesian coordinates, we have

$$\begin{aligned} I &= f\int_{x=-a/2}^{x=a/2}\left(\int_{y=-b/2}^{y=b/2}(x^2+y^2)\,dy\right)dx \\ &= f\int_{x=-a/2}^{x=a/2}\left[x^2y+\tfrac{1}{3}y^3\right]_{y=-b/2}^{y=b/2}dx \\ &= 2f\int_{-a/2}^{a/2}\left(\tfrac{1}{2}x^2b+\tfrac{1}{24}b^3\right)dx \\ &= 2f\left[\tfrac{1}{6}bx^3+\tfrac{1}{24}b^3x\right]_{-a/2}^{a/2} \\ &= \tfrac{1}{12}fab(a^2+b^2). \end{aligned}$$

Substituting in the particular values given, the moment of inertia of the ruler is

$$\begin{aligned} I &= \frac{10^{-2}\times 0.30\times 0.04}{12}(0.30^2+0.04^2) \\ &= 9.16\times 10^{-7}. \end{aligned}$$

Note that fab is the total mass M of the ruler, so we can also write

$$I = \tfrac{1}{12}M(a^2+b^2).$$

Solution to Exercise 15

Taking the constant surface density f outside the area integral, and using equation (10), we have

$$I_{\text{disc}} = f\int_{\theta=-\pi}^{\theta=\pi}\left(\int_{r=0}^{r=a}r^3\,dr\right)d\theta = 2\pi f\int_0^a r^3dr = \tfrac{1}{2}\pi fa^4.$$

The mass of the disc is $M = f\pi a^2$, so we have $I_{\text{disc}} = \frac{1}{2}Ma^2$, as required.

Solution to Exercise 16

Let f be the density of the washer. Then using equation (10) again, we have

$$I_{\text{washer}} = 2\pi f\int_a^b r^3\,dr = 2\pi f\left[\tfrac{1}{4}r^4\right]_a^b = \tfrac{1}{2}\pi f(b^4-a^4).$$

The mass of the washer is $M = f\pi(b^2-a^2)$, so $I_{\text{washer}} = \frac{1}{2}M(b^2+a^2)$.

Solution to Exercise 17

The moment of inertia of the magnifying glass is

$$\begin{aligned} I &= 2\pi k \int_0^a \left(1 - \frac{r^2}{a^2}\right) r^3 \, dr \\ &= 2\pi k \left[\frac{r^4}{4} - \frac{r^6}{6a^2}\right]_0^a \\ &= \tfrac{1}{6}\pi k a^4 = 6.1 \times 10^{-7}, \end{aligned}$$

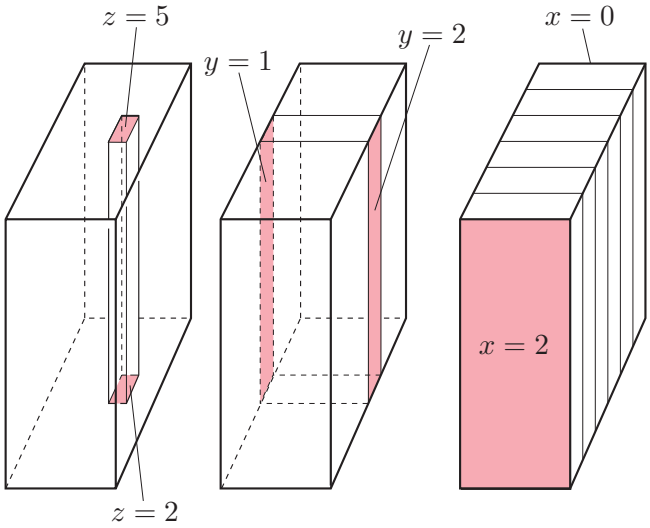
for the given values of k and a.

Solution to Exercise 18

The mass of the block is the volume integral

$$M = \int_B f \, dV = \int_B (x + y + z) \, dV.$$

The region of integration is shown in the following figure.



So the mass is

$$\begin{aligned} M &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left(\int_{z=2}^{z=5} (x + y + z) \, dz \right) dy \right) dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left[(x + y)z + \tfrac{1}{2}z^2 \right]_{z=2}^{z=5} dy \right) dx \\ &= \int_{x=0}^{x=2} \left(\int_{y=1}^{y=2} \left(3x + 3y + \tfrac{21}{2}\right) dy \right) dx \\ &= \int_{x=0}^{x=2} \left[\left(3x + \tfrac{21}{2}\right) y + \tfrac{3}{2}y^2 \right]_{y=1}^{y=2} dx \\ &= \int_{x=0}^{x=2} (3x + 15) \, dx = \left[\tfrac{3}{2}x^2 + 15x \right]_{x=0}^{x=2} = 36. \end{aligned}$$

Solution to Exercise 19

The mass of the block is

$$\begin{aligned}\int_B (x(y+1)-z)\,dV &= \int_{x=1}^{x=2}\left(\int_{y=0}^{y=3}\left(\int_{z=-1}^{z=0}(x(y+1)-z)\,dz\right)dy\right)dx\\ &= \int_{x=1}^{x=2}\left(\int_{y=0}^{y=3}\left(x(y+1)+\tfrac{1}{2}\right)dy\right)dx\\ &= \int_1^2\left(\tfrac{9}{2}x+3x+\tfrac{3}{2}\right)dx\\ &= \left[\tfrac{15}{4}x^2+\tfrac{3}{2}x\right]_1^2 = \tfrac{51}{4}.\end{aligned}$$

Solution to Exercise 20

The scalar function $f = c + \alpha r$ is spherically symmetric so, using equation (23),

$$\begin{aligned}M &= 4\pi\int_{r=0}^{r=R}(c+\alpha r)r^2\,dr\\ &= 4\pi\left[\tfrac{1}{3}cr^3+\tfrac{1}{4}\alpha r^4\right]_0^R\\ &= \tfrac{1}{3}\pi R^3(4c+3\alpha R).\end{aligned}$$

(The case $c = 1$ and $\alpha = 0$ gives the volume of a sphere.)

Solution to Exercise 21

The mass of the sphere of density D is

$$M = D\int_{r=0}^{r=R}\left(\int_{\theta=0}^{\theta=\pi}\left(\int_{\phi=-\pi}^{\phi=\pi} r^2\sin\theta\,d\phi\right)d\theta\right)dr.$$

Using equation (23), $M = \frac{4}{3}\pi DR^3$, and using this result in the solution to Example 11, we find

$$I = \tfrac{2}{5}MR^2.$$

Solution to Exercise 22

Let the constant density of the cylinder be D. The volume integral in cylindrical coordinates can be carried out in any order, so equation (21) becomes

$$I = 2\pi\int_{\rho=0}^{\rho=a}\left(\int_{z=0}^{z=h} D\,dz\right)\rho^3\,d\rho.$$

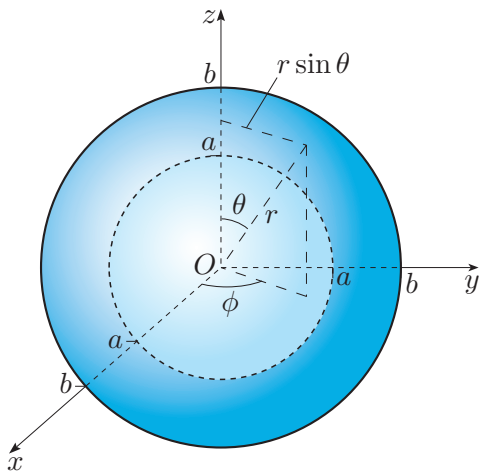
The z-integral yields a factor Dh, and the ρ-integral yields $\frac{1}{4}a^4$. Hence $I = \frac{1}{2}\pi Dha^4$. The mass M of the cylinder is given by a similar volume integral but with ρ replacing ρ^3 in the integrand. Hence $M = \pi Dha^2$, thus

$$I = \tfrac{1}{2}Ma^2.$$

Note that the volume of the cylinder is given by $M/D = \pi ha^2$.

Solution to Exercise 23

We use spherical coordinates, and the shell is as shown in the figure below.



Using equation (24) we have, with $f = D$,

$$I = D \int_{\text{shell}} (r \sin\theta)^2 r^2 \sin\theta \, d\phi \, d\theta \, dr.$$

The function to be integrated does not depend on the azimuthal angle ϕ, so the ϕ-integral yields a factor 2π. This leaves the area integral

$$2\pi D \int_{r=a}^{r=b} \left(\int_{\theta=0}^{\theta=\pi} r^4 \sin^3\theta \, d\theta \right) dr.$$

We can write $\sin^3\theta = \sin\theta(1 - \cos^2\theta)$, so the θ-integral yields

$$\left[-\cos\theta + \tfrac{1}{3}\cos^3\theta \right]_0^{\pi} = \tfrac{4}{3}.$$

Hence

$$\begin{aligned} I &= \tfrac{8}{3}\pi D \int_a^b r^4 \, dr \\ &= \tfrac{8}{15}\pi D(b^5 - a^5). \end{aligned}$$

The mass of the shell is $M = \frac{4}{3}\pi D(b^3 - a^3)$, since this is just the mass of a sphere of radius b minus that of radius a (where the mass of a sphere of radius R is given in the solution to Exercise 21), so we have

$$I = \frac{2M(b^5 - a^5)}{5(b^3 - a^3)}.$$

(Note that this answer gives $\frac{2}{5}Mb^2$ in the limit $a \to 0$, which is the moment of inertia of a uniform solid sphere of radius b as found in Exercise 21.)

Solution to Exercise 24

With the axis of the cylinder on the z-axis, and since the cylinder is uniform, the moment of inertia must be the same as the moment of inertia of the (uniform) washer in Exercise 16, since the projection of the cylindrical shell onto the (x, y)-plane occupies the same region as the washer. Thus we have

$$I_{\text{shell}} = I_{\text{washer}} = \tfrac{1}{2}M(b^2 + a^2).$$

Solution to Exercise 25

We use cylindrical coordinates with the z-axis pointing along the axis of the cylinder, and let the rotation axis be the x-axis. So the perpendicular distance d of a volume element from the rotation axis is now given by $d^2 = y^2 + z^2 = \rho^2 \sin^2\phi + z^2$. Hence with D denoting the constant density of the cylinder, and using the double-angle formula $\sin^2\phi = (1 - \cos 2\phi)/2$, we have

$$\begin{aligned}
I &= \int_{\text{cylinder}} d^2 D \, dV \\
&= D \int_{\text{cylinder}} \left(\rho^2 \sin^2\phi + z^2\right) dV \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(\int_{z=-L/2}^{z=L/2} \left(\rho^2 \sin^2\phi + z^2\right) \rho \, dz \right) d\phi \right) d\rho \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left[\rho^2 z \sin^2\phi + \tfrac{1}{3}z^3\right]_{z=-L/2}^{z=L/2} \rho \, d\phi \right) d\rho \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(L\rho^2 \sin^2\phi + \tfrac{1}{12}L^3\right) \rho \, d\phi \right) d\rho \\
&= D \int_{\rho=0}^{\rho=R} \left(\int_{\phi=-\pi}^{\phi=\pi} \left(L\rho^2 \tfrac{1}{2}(1 - \cos 2\phi) + \tfrac{1}{12}L^3\right) \rho \, d\phi \right) d\rho \\
&= \pi D \int_0^R \left(L\rho^3 + \tfrac{1}{6}L^3\rho\right) d\rho \\
&= \pi D \left(\tfrac{1}{4}LR^4 + \tfrac{1}{12}L^3R^2\right) \\
&= \tfrac{1}{12}\pi D L R^2 (3R^2 + L^2).
\end{aligned}$$

Now recall that the volume of a cylinder of length L and radius R is given by $\pi R^2 L$ (see Exercise 22), so the density is $D = M/(\pi R^2 L)$. Thus substituting for D in the above expression for I gives the required result:

$$I = \tfrac{1}{12}M(3R^2 + L^2).$$

Note that in the limit as $R \to 0$, this becomes $\frac{1}{12}ML^2$, the moment of inertia of a thin rod of length L about an axis through its centre perpendicular to its length, as found in Subsection 2.3.

Solution to Exercise 26

(a) We use spherical coordinates. The density function is $f(\mathbf{r}) = cr^2$, so using equation (23), the mass of the sphere is

$$M = 4\pi \int_0^R (cr^2) r^2 \, dr = \tfrac{4}{5}\pi c R^5.$$

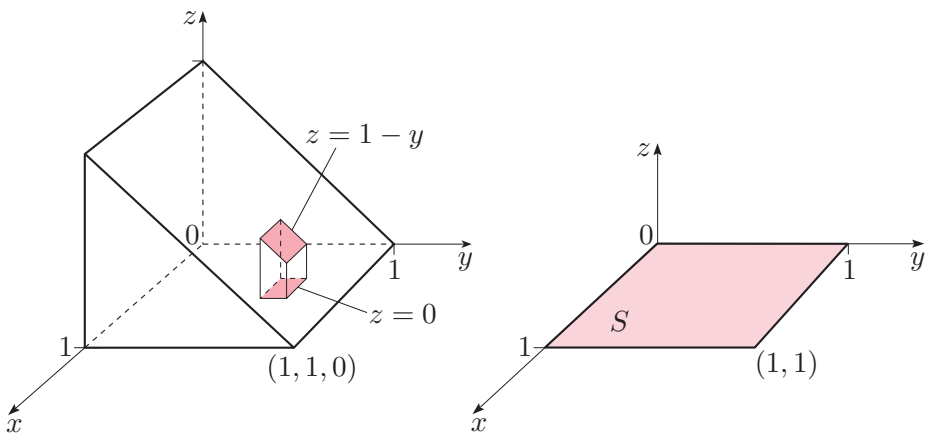
(b) The moment of inertia I is the volume integral of the function $g(r, \theta, \phi) = f(r)\, d^2 = (cr^2)(r \sin\theta)^2$ (which is *not* spherically symmetric). Hence

$$\begin{aligned} I &= \int_{\text{sphere}} cr^2 (r\sin\theta)^2 \, dV \\ &= c \int_{\text{sphere}} r^2 (r\sin\theta)^2 r^2 \sin\theta \, d\phi \, d\theta \, dr \\ &= c \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=\pi} \int_{\phi=-\pi}^{\phi=\pi} r^6 \sin^3\theta \, d\phi \, d\theta \, dr \\ &= c \times 2\pi \times \tfrac{4}{3} \times \tfrac{1}{7} R^7 = \tfrac{8}{21} c\pi R^7 = \tfrac{10}{21} M R^2, \end{aligned}$$

using $\int_0^\pi \sin^3\theta \, d\theta = \frac{4}{3}$ from the solution to Exercise 23 or Example 11.

Solution to Exercise 27

The region and the projection onto the (x, y)-plane are shown below.



The integral over z gives

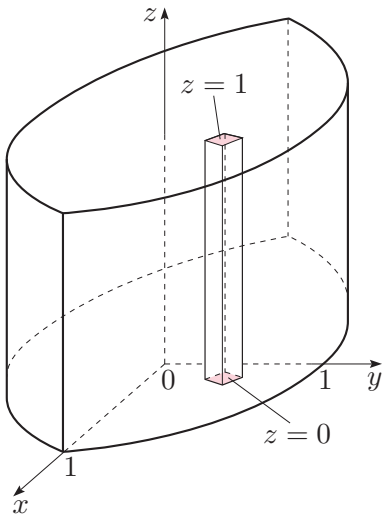
$$\begin{aligned} h(x, y) &= \int_{z=0}^{z=1-y} x^2 yz \, dz \\ &= \left[\tfrac{1}{2} x^2 y z^2\right]_{z=0}^{z=1-y} = \tfrac{1}{2} x^2 y (1-y)^2. \end{aligned}$$

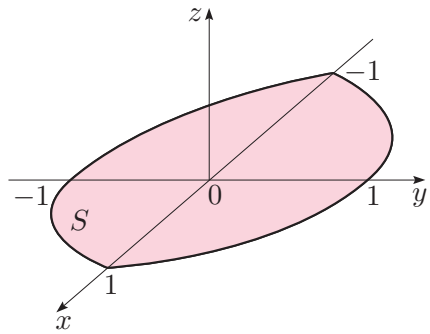
Now following the steps in Procedure 1 for evaluating the area integral of $h(x, y)$ over S, we have

$$\begin{aligned}\int_S h(x,y)\,dA &= \int_{x=0}^{x=1}\left(\int_{y=0}^{y=1} \tfrac{1}{2}x^2y(1-y)^2\,dy\right)dx \\ &= \int_{x=0}^{x=1}\left[\tfrac{1}{2}x^2\left(\tfrac{1}{2}y^2-\tfrac{2}{3}y^3+\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1}dx \\ &= \int_0^1 \tfrac{1}{24}x^2\,dx = \tfrac{1}{72}.\end{aligned}$$

Solution to Exercise 28

The region and its projection onto the (x, y)-plane (bounded by the parabolas $y = 1 - x^2$ and $y = x^2 - 1$) are illustrated below.

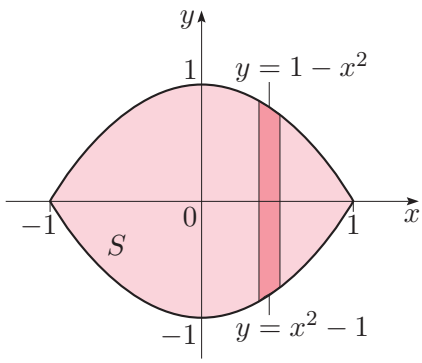




The column is shown in the left-hand figure. The z-limits are $z = 0$ and $z = 1$. The integral over z is

$$\begin{aligned}h(x,y) &= \int_{z=0}^{z=1}(z+3x-2)\,dz \\ &= \left[\tfrac{1}{2}z^2+(3x-2)z\right]_{z=0}^{z=1} = \tfrac{1}{2}+3x-2 = 3x-\tfrac{3}{2}.\end{aligned}$$

We now evaluate the area integral over the shape shown below.



So we have

$$\begin{aligned}\int_S h(x,y)\,dA &= \int_{x=-1}^{x=1}\left(\int_{y=x^2-1}^{y=1-x^2}\left(3x-\tfrac{3}{2}\right)dy\right)dx\\ &= 3\int_{-1}^{1}(2x-1)(1-x^2)\,dx\\ &= 3\left[x^2-\tfrac{1}{2}x^4-x+\tfrac{1}{3}x^3\right]_{-1}^{1}\\ &= -4.\end{aligned}$$

Solution to Exercise 29

(a) Use cylindrical coordinates and note that the curved boundary of the paraboloid is given by the equation $z = h\rho^2/a^2$, which can also be expressed as $\rho = a\sqrt{z/h}$. The volume is given by the volume integral $\int_B 1\,dV$, which is evaluated in cylindrical coordinates by performing the ρ-integral before the z-integral, that is, the opposite order to Example 13, as this leads to slightly easier single integrals. Also, since the integrand and the boundary of B are independent of ϕ, the ϕ-integral can be evaluated independently of the z- and ρ-integrals, leading to an overall factor of 2π. Thus we have

$$\begin{aligned}\int_B 1\,dV &= 2\pi\int_{z=0}^{z=h}\left(\int_{\rho=0}^{\rho=a\sqrt{z/h}}\rho\,d\rho\right)dz\\ &= \frac{\pi a^2}{h}\int_0^h z\,dz\\ &= \tfrac{1}{2}\pi a^2 h.\end{aligned}$$

(b) Let the density of the uniform solid be D. Then

$$D = M/\int_B 1\,dV = 2M/(\pi a^2 h).$$

The moment of inertia about the z-axis is given by

$$I = \int_B \rho^2 D\,dV,$$

and again, since the integrand and boundary of B are independent of ϕ, we have

$$\begin{aligned}I &= 2\pi\int_{z=0}^{z=h}\left(\int_{\rho=0}^{\rho=a\sqrt{z/h}}\rho^3 D\,d\rho\right)dz\\ &= 2\pi D\int_{z=0}^{z=h}\left(\left[\tfrac{1}{4}\rho^4\right]_{\rho=0}^{\rho=a\sqrt{z/h}}\right)dz\\ &= \tfrac{1}{2}\pi D\frac{a^4}{h^2}\int_0^h z^2\,dz\\ &= \tfrac{1}{6}\pi D a^4 h.\end{aligned}$$

Substituting for D then leads directly to $I = \frac{1}{3}Ma^2$, as required.

Solution to Exercise 30

(a) We use cylindrical coordinates and integrate over z first as was done in Example 13. The curved surface of the spherical cap is given by the equation $\rho^2 + z^2 = R^2$. Thus for a given ρ, the upper and lower limits of the z-integral are $\sqrt{R^2 - \rho^2}$ and $R/2$, respectively. Moreover, the largest value of ρ, that is, the radius of the circular base of the cap, is given by $\sqrt{R^2 - (R/2)^2} = \sqrt{3}R/2$. Hence the volume of the spherical cap is given by

$$\begin{aligned}\int_{\text{spherical cap}} 1\,dV &= \int_{\rho=0}^{\rho=\sqrt{3}R/2}\left(\int_{\phi=-\pi}^{\phi=\pi}\left(\int_{z=R/2}^{z=\sqrt{R^2-\rho^2}}\rho\,dz\right)d\phi\right)d\rho\\ &= 2\pi\int_0^{\sqrt{3}R/2}\left(\sqrt{R^2-\rho^2}-\tfrac{1}{2}R\right)\rho\,d\rho\\ &= 2\pi\left[-\tfrac{1}{3}(R^2-\rho^2)^{3/2}-\tfrac{1}{4}R\rho^2\right]_0^{\sqrt{3}R/2}\\ &= 2\pi\left(\tfrac{1}{3}-\tfrac{1}{3}\left(\tfrac{1}{4}\right)^{3/2}-\tfrac{3}{16}\right)R^3\\ &= \tfrac{5}{24}\pi R^3.\end{aligned}$$

(b) Let D denote the density of the solid. Then the moment of inertia I of the spherical cap about the z-axis (axis of symmetry) is given by

$$\begin{aligned}I &= \int_{\text{spherical cap}}\rho^2 D\,dV\\ &= \int_{\rho=0}^{\rho=\sqrt{3}R/2}\left(\int_{\phi=-\pi}^{\phi=\pi}\left(\int_{z=R/2}^{z=\sqrt{R^2-\rho^2}}D\rho^3\,dz\right)d\phi\right)d\rho\\ &= 2\pi D\int_0^{\sqrt{3}R/2}\left(\sqrt{R^2-\rho^2}-\tfrac{1}{2}R\right)\rho^3\,d\rho.\end{aligned}$$

Using the hint, we can write

$$\begin{aligned}I &= 2\pi D\left[\tfrac{1}{5}(R^2-\rho^2)^{5/2}-\tfrac{1}{3}R^2(R^2-\rho^2)^{3/2}-\tfrac{1}{8}R\rho^4\right]_0^{\sqrt{3}R/2}\\ &= 2\pi DR^5\left(\tfrac{1}{5}\left(\tfrac{1}{4}\right)^{5/2}-\tfrac{1}{3}\left(\tfrac{1}{4}\right)^{3/2}-\tfrac{9}{128}-\tfrac{1}{5}+\tfrac{1}{3}\right)\\ &= \tfrac{53}{960}\pi DR^5.\end{aligned}$$

The density D is the mass per unit volume. Hence

$$D = M/\int_{\text{spherical cap}} 1\,dV = 24M/(5\pi R^3),$$

and substituting this into the above expression for I gives

$$I = \frac{53\pi}{960}\times\frac{24M}{5\pi R^3}\times R^5 = \frac{53}{200}MR^2,$$

as required.

Solution to Exercise 31

This is the same surface as in Example 14, but over a different region of the (x, y)-plane. Thus $g(x, y) = \sqrt{1 + x^2 + y^2}$ and

$$\Sigma = \int_D \sqrt{1 + x^2 + y^2}\, dA,$$

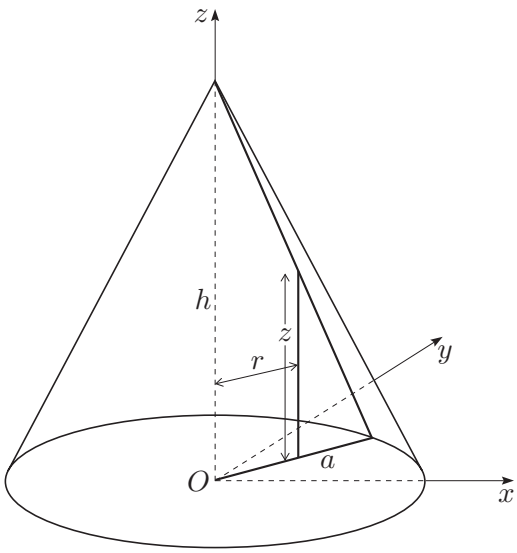
where D denotes the disc centred at the origin with radius a. Given the shape of the region D, the area integral is best evaluated using polar coordinates, noting that $\sqrt{1 + x^2 + y^2} = \sqrt{1 + r^2}$, which depends *only* on the radial coordinate r. Thus, from equation (10) (Subsection 2.2), we have

$$\Sigma = 2\pi \int_0^a \sqrt{1 + r^2}\, r\, dr = \tfrac{2}{3}\pi\big[(1 + r^2)^{3/2}\big]_0^a = \tfrac{2}{3}\pi\big((1 + a^2)^{3/2} - 1\big).$$

Again, we have used the trick given in equation (6), with $k = \frac{1}{2}$, $n = \frac{1}{2}$ and $f(r) = 1 + r^2$.

Solution to Exercise 32

A diagram of the cone is shown below.



The equation of the cone can easily be found by using similar triangles. Since $r = \sqrt{x^2 + y^2}$, we have

$$\frac{z}{a - r} = \frac{h}{a}.$$

Thus $z = h - hr/a$, or

$$z = h - \frac{h}{a}\sqrt{x^2 + y^2}.$$

The partial derivatives are therefore

$$\frac{\partial z}{\partial x} = \frac{-hx}{a\sqrt{x^2+y^2}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{-hy}{a\sqrt{x^2+y^2}},$$

and the area of the cone is given by

$$\begin{aligned}\Sigma &= \int_S \sqrt{1 + \frac{h^2x^2}{a^2(x^2+y^2)} + \frac{h^2y^2}{a^2(x^2+y^2)}}\, dA \\ &= \int_S \sqrt{\frac{a^2+h^2}{a^2}}\, dA,\end{aligned}$$

where S is a disc of radius a lying in the (x, y)-plane.

Note that the integrand $\sqrt{(a^2+h^2)/a^2}$ is constant throughout the region S, therefore we can write

$$\Sigma = \frac{\sqrt{a^2+h^2}}{a} \int_S 1\, dA.$$

But $\int_S 1\, dA$ is just the area πa^2 of region S. Hence

$$\Sigma = \frac{\sqrt{a^2+h^2}}{a} \times \pi a^2 = \pi a\sqrt{a^2+h^2}.$$

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